

# UPWIND DIFFERENCE APPROXIMATIONS FOR DEGENERATE PARABOLIC CONVECTION-DIFFUSION EQUATIONS WITH A DISCONTINUOUS COEFFICIENT

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**ABSTRACT.** We analyze approximate solutions generated by an upwind difference scheme (of Engquist-Osher type) for nonlinear degenerate parabolic convection-diffusion equations where the nonlinear convective flux function has a discontinuous coefficient  $\gamma(x)$  and the diffusion function  $A(u)$  is allowed to be strongly degenerate (the pure hyperbolic case is included in our setup). The main problem is obtaining a uniform bound on the total variation of the difference approximation  $u^\Delta$ , which is a manifestation of resonance. To circumvent this analytical problem, we construct a singular mapping  $\Psi(\gamma, \cdot)$  such that the total variation of the transformed variable  $z^\Delta = \Psi(\gamma^\Delta, u^\Delta)$  can be bounded uniformly in  $\Delta$ . This establishes strong  $L^1$  compactness of  $z^\Delta$  and, since  $\Psi(\gamma, \cdot)$  is invertible, also  $u^\Delta$ . Our singular mapping is novel in that it incorporates a contribution from the diffusion function  $A(u)$ . We then show that the limit of a converging sequence of difference approximations is a weak solution as well as satisfying a Kruzkov-type entropy inequality. We prove that the diffusion function  $A(u)$  is Hölder continuous, implying that the constructed weak solution  $u$  is continuous in those regions where the diffusion is nondegenerate. Finally, some numerical experiments are presented and discussed.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

We are interested in upwind finite difference approximations for nonlinear degenerate parabolic convection-diffusion initial value problems of the type

$$(1.1) \quad \begin{cases} u_t + f(\gamma(x), u)_x = A(u)_{xx}, & (x, t) \in \Pi_T = \mathbf{R} \times (0, T), \\ u(x, 0) = u_0(x), & x \in \mathbf{R}, \end{cases}$$

where  $T > 0$  is fixed,  $u(x, t)$  is the scalar unknown function that is sought, and  $f, \gamma, A, u_0$  are given functions to be detailed later. The special feature of the problem studied herein, which makes mathematical and numerical analysis more complicated, is the combination of a convection part that depends explicitly on the spatial location through a coefficient  $\gamma(x)$  that may be *discontinuous* and a diffusion part that *strongly degenerates* in the sense that  $A'(\cdot) \geq 0$ . In fact, included in our setup are hyperbolic conservation laws with a discontinuous coefficient:

$$(1.2) \quad u_t + f(\gamma(x), u)_x = 0.$$

To facilitate the analysis, (1.2) is often written as a  $2 \times 2$  nonstrictly hyperbolic system of equations:

$$(1.3) \quad \gamma_t = 0, \quad u_t + f(\gamma, u)_x = 0.$$

Problems of the type (1.1) occur in several applications. Biased by our own interests, we mention here only flow in porous media (see, e.g., [8, 14]) and sedimentation-consolidation processes [4, 5]. The purely convective version of (1.1) ( $A'(u) \equiv 0$ ) provides a simple model of traffic flow on a highway [49, 25], the spatially varying coefficient  $\gamma$  corresponding to varying road conditions. Scalar

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conservation laws with discontinuities in the flux also arise in radar shape-from-shading problems [41] and as building blocks in dimensional splitting methods for multi-dimensional Hamilton-Jacobi equations [30].

Before continuing, let us detail the assumptions that we need to impose on the “data” of the problem (1.1). For the coefficient  $\gamma$ , we assume that

$$\gamma(x) \in [\underline{\gamma}, \bar{\gamma}] \quad \forall x \in \mathbf{R}; \quad \gamma \in BV(\mathbf{R}).$$

In particular,  $\gamma$  is allowed to be discontinuous. For the convective flux function  $f$ , we assume that

$$(1.4) \quad f(\gamma, 0) = f_0 \in \mathbf{R} \text{ for all } \gamma \text{ and } f(\gamma, 1) = f_1 \in \mathbf{R} \text{ for all } \gamma.$$

The purpose of this assumption is to guarantee that a solution initially in the interval  $[0, 1]$  remains in  $[0, 1]$  for all subsequent times. Furthermore, we assume that  $f \in \text{Lip}([\underline{\gamma}, \bar{\gamma}] \times [0, 1])$ . With this assumption the partial derivatives  $f_\gamma$  and  $f_u$  exist almost everywhere, and  $\|f_\gamma\|_\infty$  and  $\|f_u\|_\infty$  are Lipschitz constants of  $f$  with respect to  $\gamma$  and  $u$ . In what follows, we will often use the notational shorthands  $\|f_u\|$ ,  $\|f_\gamma\|$  instead of  $\|f_u\|_\infty$ ,  $\|f_\gamma\|_\infty$ , respectively. Let

$$f_u^+(\gamma, u) = \max(0, f_u(\gamma, u)), \quad f_u^-(\gamma, u) = \min(0, f_u(\gamma, u)).$$

We will also require the technical assumption that  $f_u$  is Lipschitz continuous as a function of  $\gamma$ , with Lipschitz constant  $L_{u\gamma}$ . It follows that  $f_u^-$  and  $f_u^+$  are also Lipschitz continuous in  $\gamma$  with the same Lipschitz constant. For example, if  $f(\gamma, u) = \gamma \tilde{f}(u)$ , where  $\tilde{f} \in \text{Lip}([0, 1])$ , this Lipschitz assumption will hold with  $L_{u\gamma} = \|\tilde{f}'\|$ . We assume that for each  $\gamma \in [\underline{\gamma}, \bar{\gamma}]$ , there is a unique maximum  $u^*(\gamma) \in [0, 1]$  such that  $f(\gamma, \cdot)$  is strictly increasing for  $u < u^*(\gamma)$  and strictly decreasing for  $u > u^*(\gamma)$ , and that for each fixed  $\gamma \in [\underline{\gamma}, \bar{\gamma}]$ ,  $|f_u(\gamma, u)| > 0$  for almost all  $u \in [0, 1]$ . Finally, we also assume that  $u^*$  is Lipschitz continuous as a function of  $\gamma$ .

Regarding the diffusion function  $A$ , we assume that it belongs to  $\text{Lip}([0, 1])$  with Lipschitz constant  $\|A'\|$  and that the following *degenerate parabolicity* condition holds:

$$(1.5) \quad A(\cdot) \text{ is nondecreasing with } A(0) = 0.$$

Actually we shall be a bit more precise than (1.5). We assume that  $A$  degenerates (i.e., is constant) on a finite set of disjoint intervals, that is,

$$A'(w) = 0, \quad \forall w \in \bigcup_{i=1}^M [\alpha_i, \beta_i] := \Omega,$$

where  $\alpha_i < \beta_i$ ,  $i = 1, \dots, M$ ,  $M \geq 1$ . On these intervals, (1.1) acts as a pure hyperbolic problem. We assume that  $A$  is non-degenerate (i.e., strictly increasing) off these intervals, so that (1.1) acts as a parabolic problem on  $[0, 1] \setminus \Omega$ . It is assumed that the maximum  $u^*(\gamma)$  either lies in  $\Omega$  for all  $\gamma$ , or lies in the closure of  $[0, 1] \setminus \Omega$  for all  $\gamma$ . This is trivially satisfied, for example, if  $f(\gamma, u) = \gamma \tilde{f}(u)$ , since then  $u^*$  is constant. In view of (1.5), one often refers to (1.1) as a mixed hyperbolic-parabolic problem. In what follows, we assume (since the pure hyperbolic case has already been treated in [44, 45])

$$\max_{w \in [0, 1]} A'(w) > 0.$$

Finally, we assume that the initial function  $u_0$  satisfies

$$(1.6) \quad \begin{cases} u_0 \in L^1(\mathbf{R}) \cap BV(\mathbf{R}); & u_0(x) \in [0, 1] \quad \forall x \in \mathbf{R}; \\ A(u_0) \text{ is absolutely continuous on } \mathbf{R}; \\ A(u_0)_x \in BV(\mathbf{R}). \end{cases}$$

Our assumption that  $A(u_0)$  is absolutely continuous requires that any jump in  $u_0$  must be contained within one of the intervals  $[\alpha_i, \beta_i]$  where  $A$  is constant.

Independently of the smoothness of  $\gamma$ , if (1.1) is allowed to degenerate at certain points, that is,  $A'(s) = 0$  for some values of  $s$ , solutions are not necessarily smooth and weak solutions must be sought. A *weak solution* is here defined as follows:

**Definition 1.1.** *A measurable function  $u(x, t)$  is a weak solution of the initial value problem (1.1) if it satisfies the following conditions:*

(D.1)  $u \in L^1(\Pi_T) \cap L^\infty(\Pi_T) \cap C(0, T; L^1(\mathbf{R}))$  and  $A(u) \in L^2(0, T; H^1(\mathbf{R}))$ .

(D.2) For all test functions  $\phi \in \mathcal{D}(\Pi_T)$  such that  $\phi|_{t=T} = 0$ ,

$$(1.7) \quad \iint_{\Pi_T} \left( u\phi_t + (f(\gamma(x), u) - A(u)_x)\phi_x \right) dt dx + \int_{\mathbf{R}} u_0(x)\phi(x, 0) = 0.$$

Solutions behave even more dramatically if  $A'(s)$  is zero on a whole interval  $[\alpha, \beta]$ . Then (weak) solutions may be discontinuous and they are not uniquely determined by their initial data. Consequently, an entropy condition must be imposed to single out the physically correct solution. If  $\gamma$  is “smooth”, a weak solution  $u$  satisfies the *entropy condition* if for all convex  $C^2$  functions  $\eta : \mathbf{R} \rightarrow \mathbf{R}$ ,

$$(1.8) \quad \eta(u)_t + (q(\gamma(x), u))_x + r(u)_{xx} + \gamma'(x)(\eta'(u)f_\gamma(\gamma(x), u) - q_\gamma(\gamma(x), u)) \leq 0 \text{ in } \mathcal{D}'(\Pi_T),$$

where  $q, r : \mathbf{R} \rightarrow \mathbf{R}$  are defined by

$$q_u(\gamma(x), u) = \eta'(u)f_u(\gamma(x), u), \quad r'(u) = \eta'(u)A'(u).$$

By a standard limiting argument, (1.8) implies that the Kruřkov-type entropy condition

$$(1.9) \quad |u - c|_t + [\text{sign}(u - c)(f(\gamma(x), u) - f(\gamma(x), c))]_x + |A(u) - A(c)|_{xx} + \gamma'(x)\text{sign}(u - c)f_\gamma(\gamma(x), c) \leq 0$$

holds in  $\mathcal{D}'(\Pi_T)$  for all  $c \in \mathbf{R}$ . The entropy condition (1.9) goes back to Kruřkov [37], Vol’pert [46], and Vol’pert and Hudjaev [48]. Existence, uniqueness and stability results for entropy solutions of strongly degenerate parabolic equations with smooth coefficients can be found in [2, 6, 9, 28, 50, 48, 47]. For example, when the coefficients are sufficiently smooth and the initial function satisfies (1.6), there exists a unique entropy solution of (1.1) that belongs to  $BV(\Pi_T)$  (i.e.,  $u_x$  and  $u_t$  are finite measures on  $\Pi_T$ ) and  $A(u)$  belongs to the Hölder space  $C^{1, \frac{1}{2}}(\Pi_T)$ .

The entropy solution theory breaks down when  $\gamma$  is discontinuous. In Karlsen, Risebro, and Towers [31], we took a first step towards analyzing degenerate parabolic equations with a discontinuous coefficient. More precisely, we proved existence of a weak solution by passing to the limit in a problem where we had smoothed out the coefficient and added artificial viscosity. In contrast to the present paper, the convergence proof in [31] used the compensated compactness theory.

In this paper, we are interested in constructing a “simple” numerical scheme for (1.1) and proving its strong convergence towards a weak solution. When  $\gamma$  is constant or at least smooth, several numerical schemes have been proposed and analyzed already in the literature. Let us mention the operator splitting methods in [16, 24], the finite difference schemes in [18, 15, 17], the finite volume schemes in [1, 40, 20], and the kinetic BGK schemes in [3]. For a partial overview of mathematical and numerical theory for degenerate parabolic equations based on “hyperbolic” techniques, see [14].

We now present the numerical scheme that we propose for (1.1) when  $\gamma$  is possibly discontinuous. Let  $\Delta x > 0$  and  $\Delta t > 0$  denote the spatial and temporal discretization parameters respectively. We then let  $U_j^n$  denote the finite difference approximation of  $u(j\Delta x, n\Delta t)$ . The difference scheme, which uses the Engquist-Osher numerical flux [13] for the convection part and centered differencing for the parabolic part, takes the following (conservation) form

$$(1.10) \quad \frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) - h(\gamma_{j-\frac{1}{2}}, U_j^n, U_{j-1}^n)}{\Delta x} = \frac{A(U_{j+1}^n) - 2A(U_j^n) + A(U_{j-1}^n)}{(\Delta x)^2}.$$

Here the numerical flux  $h$  is the Engquist-Osher generalized upwind flux [13] (see Section 3 for precise statements). The scheme (1.10) is the one-dimensional version of the multidimensional algorithm presented in [29], where convergence was established for a “rough” but continuous coefficient  $\gamma$ . It is also closely related to the algorithm presented in [44] and [45], where a staggered mesh EO scheme was investigated for a purely hyperbolic problem with a discontinuous coefficient.

In particular, the discretization of  $\gamma$  is staggered with respect to that of the conserved variable  $u$ . This results in a significant reduction in complexity compared with the alternative of aligning the two discretizations. In the latter case, a more complicated  $2 \times 2$  Riemann problem has to be solved (exactly or approximately) [38, 39, 21, 35, 34]. Staggering the discretizations also greatly simplifies the analysis, making it possible to apply, with some allowances for the parabolic terms, some of the analytical techniques developed for monotone difference schemes for purely hyperbolic problems. Another important feature of our scheme is its conservation form, i.e., it is a shock capturing algorithm in the purely hyperbolic regime where  $A' = 0$ . For the case of constant  $\gamma$ , [18] provides numerical evidence that differencing the PDE (1.1) directly (i.e., not in conservation form) results in wrong solutions, specifically shocks may move with the wrong speed. Finally, our algorithm is a so-called upwind scheme, meaning that the differencing of the convective flux is biased in the direction of incoming waves, making it possible to resolve shocks without excessive smearing.

Let  $u^\Delta(x, t)$  be the piecewise constant approximate solution generated by (1.10), and define  $\sigma(w) = w/|w|$  if  $w \neq 0$  and  $\sigma(0) = 0$ . Roughly speaking, our main results can be stated as follows

**Theorem 1.1.** *We have that  $u^\Delta$  converges along a subsequence in  $L^1_{\text{loc}}(\Pi_T)$  to a weak solution  $u$  of the initial value problem (1.1) in the sense of Definition 1.1. Furthermore, if  $\gamma$  has finitely many discontinuities located at  $\xi_1, \dots, \xi_{M'}$ , the limit satisfies the following entropy condition in  $\mathcal{D}'(\Pi_T)$  for all  $c \in \mathbf{R}$ :*

$$(1.11) \quad |u - c|_t + (\sigma(u - c)(f(\gamma(x), u) - f(\gamma(x), c)))_x + |A(u) - A(c)|_{xx} - |f(\gamma(x), c)_x| \leq 0.$$

The convergence proof consists of establishing bounds on the solution and its  $L^1$  space and time translates, measured with respect to a transformed variable  $z$ . Specifically, we prove that the scalar upwind difference scheme converges (along a subsequence) to a weak solution of (1.1) by constructing a singular mapping  $\Psi : (\gamma, u) \mapsto (\gamma, z)$  such that strong compactness of  $z^\Delta = \Psi(\gamma^\Delta, u^\Delta)$  can be obtained. As in other problems concerning resonance phenomena, it is necessary to measure the space translates with respect to a nonlinear transformation  $\Psi$ , since there is generally no spatial variation bound for the conserved variable  $u$  itself. The singular mapping approach has been used for at least twenty years in the purely hyperbolic setting. However, the presence of the parabolic term in (1.1) requires a novel, and somewhat more complicated, singular mapping. Specifically, the singular mapping (3.1) includes a contribution from the diffusion term  $A(u)$ , which make the subsequent analysis a bit more intricate than in the purely hyperbolic case. We prove compactness for two separate parts of the singular mapping. One part,  $\mathcal{F}(\gamma, u)$ , is associated with the convective portion of the problem, and the other,  $A(u)$ , is associated with the diffusive portion of the problem. In the process of establishing compactness for the diffusive portion, we also prove that the limit  $u$  satisfies  $A(u) \in L^2(0, T; H^1(\mathbf{R}))$ . We then combine the two portions to recover the original singular mapping  $\mathcal{F}(\gamma, u) + A(u)$ , and conclude that since the mapping is strictly increasing as a function of the conserved variable  $u$ , convergence of the transformed variable implies convergence of  $u$ . We also establish regularity of the diffusion function  $A(u)$ , specifically that  $A(u) \in C^{1, \frac{1}{2}}(\Pi_T)$ , proving that the solution  $u$  itself is continuous in the regions where there is nonzero diffusion.

For the purely hyperbolic problem, the singular mapping approach can be traced back to Temple [43], who originated the technique in order to establish convergence of the Glimm scheme for a  $2 \times 2$  resonant system of conservation laws modeling the displacement of oil in a reservoir by water and polymer. In addition to the Glimm scheme, convergence has been established for the  $2 \times 2$  Godunov method by Lin, Temple, and Wang [38, 39]. Specifically, they applied the  $2 \times 2$  Godunov method to the system (1.3) and used a version of the singular mapping to establish compactness (see also Hong [26] for an “improved” singular mapping). The front tracking method, which is based on the work of Dafermos [11] and Holden, Holden, and Høegh-Krohn [23], has been applied to a number of hyperbolic problems with discontinuous coefficients. Gimse and Risebro [21] used the front tracking method to study the two phase flow equation, proving compactness of the sequence of approximations via a bound on the spatial variation, measured with respect to the singular mapping. For the scalar conservation law with a concave flux, Klingenberg and

Risebro [35] used the front tracking technique to establish existence, uniqueness, and asymptotic behavior for the Cauchy problem (1.3). The front tracking method has also been applied to the situation where the flux  $f$  is neither concave nor convex [34]. A version of the singular mapping was used in both [35] and [34], see also [33]. The singular mapping has also been used to establish convergence of difference schemes for scalar conservation laws having a discontinuous coefficient, see Towers [44, 45]. In [44], it was also proved that limits of the difference approximations satisfied a Kruřkov-type entropy condition. Moreover, uniqueness was established in the class of piecewise smooth weak solutions satisfying this entropy condition. This latter result is generalized to (1.1) in [32].

The following example, due to Lin, Temple, and Wang [38], helps to understand the impact of resonance in the purely hyperbolic setting. Assuming that the flux is smooth, they linearize the conservation law. Focusing on the case (for the sake of simplicity) where the original equation is  $u_t + (\gamma(x)f(u))_x = 0$ , the version that results from linearizing about  $u^*$  (where  $f'(u^*) = 0$ ) is

$$u_t + f(u^*)\gamma_x = 0,$$

which has the solution  $u(x, t) = -f(u^*)\gamma_x t + u_0(x)$ . If  $\gamma$  belongs to  $C^1$ , this solution, along with its variation, grows linearly with time, i.e., neither the solution nor its variation is bounded. Furthermore, if  $\gamma$  is allowed to have jumps, then the solution only makes sense as a measure. If one instead linearizes about a point where  $f'(u) \neq 0$  and the initial data  $u_0$  is bounded, both the solution and its variation will remain bounded; this follows from [45] and Proposition 2.1 herein.

The present paper provides the groundwork for future work in several directions. The parabolic term forces a very small time step on our explicit scheme, and so we intend to investigate an implicit version, which will allow for a more efficient algorithm. We also plan to present a second order version of the scheme based on using flux limiters in a novel way that keeps the total variation bounded, as measured via the singular function. We will generalize to the situation where the diffusion term varies spatially, and incorporate more general invariant regions, making it possible to relax the condition (1.4), and allow for singular source terms. Additionally, we will prove uniqueness for piecewise smooth solutions of the initial value problem (1.1) satisfying the Kruřkov-type entropy inequality (1.11). One more avenue of investigation is to relax the condition that the flux have a single maximum, allowing for any finite number of critical points.

The rest of this paper is organized as follows: Section 2 provides preliminary material concerning the definition of our algorithm and the resulting approximate solutions. In Section 3 we state and prove our main result, convergence of the scheme (2.4) to a weak solution of the initial value problem (1.1). Section 4 establishes that the diffusion function  $A(u)$  is continuous of class  $C^{1, \frac{1}{2}}$ . In Section 5 we demonstrate that our scheme satisfies a cell entropy inequality, and that as a consequence, piecewise smooth limits of our algorithm satisfy a Kruřkov-type entropy inequality. Section 6 provides the results of some numerical experiments.

## 2. DEFINITION OF APPROXIMATE SOLUTIONS

Let  $\Delta x > 0$  and  $\Delta t > 0$  be the spatial and temporal discretization parameters. The spatial domain  $\mathbf{R}$  is discretized into cells

$$I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}),$$

where  $x_k = k\Delta x$  for  $k = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots$ . Similarly, the time interval  $[0, T]$  is discretized via  $t_n = n\Delta t$  for  $n = 0, \dots, N$ , where the integer  $N$  is chosen such that  $N\Delta t = T$ , resulting in the time strips

$$I^n = [t_n, t_{n+1}).$$

We let  $\chi_j(x)$  and  $\chi^n(t)$  be the characteristic functions for the intervals  $I_j$  and  $I^n$ , respectively. We let  $\chi_j^n(x, t) = \chi_j(x)\chi^n(t)$  be the characteristic function for the rectangle

$$R_j^n = I_j \times I^n.$$

Also, we let  $R_{j+\frac{1}{2}}^n = I_{j+\frac{1}{2}} \times [t_n, t_{n+1})$  with  $I_{j+\frac{1}{2}} = [x_j, x_{j+1})$ . To simplify the presentation, we use  $\Delta_+$  and  $\Delta_-$  to designate the difference operators in the  $x$  direction, e.g.,

$$\Delta_+ f(\gamma_j, U_j^n) = f(\gamma_{j+1}, U_{j+1}^n) - f(\gamma_j, U_j^n) = \Delta_- f(\gamma_{j+1}, U_{j+1}^n).$$

Furthermore,  $\Delta_+^u$  and  $\Delta_-^u$  are spatial difference operators with respect to  $u$  only, keeping  $\gamma$  fixed, e.g.,

$$\Delta_+^u f(\gamma_j, U_j^n) = f(\gamma_j, U_{j+1}^n) - f(\gamma_j, U_j^n).$$

Before we can state the finite difference scheme, we need to introduce the Engquist-Osher (EO henceforth) numerical flux [13]:

$$h(\gamma, v, u) = \frac{1}{2}(f(\gamma, u) + f(\gamma, v)) - \frac{1}{2} \int_u^v |f_u(\gamma, w)| dw.$$

The EO numerical flux is consistent with the actual flux in the sense that

$$h(\gamma, u, u) = f(\gamma, u).$$

In addition, for fixed  $\gamma$ ,  $h(\gamma, v, u)$  is a two-point monotone flux, meaning that it is nonincreasing with respect to  $v$ , and nondecreasing with respect to  $u$ . Due to the regularity assumptions about the flux  $f$ , the numerical flux  $h$  is Lipschitz continuous with respect to each of its arguments, and in fact satisfies

$$(2.1) \quad f_u^-(\gamma, v) = h_v(\gamma, v, u) \leq 0 \leq h_u(\gamma, v, u) = f_u^+(\gamma, u).$$

Thus, if the flux  $f$  is  $C^1$  smooth, the numerical flux is also  $C^1$  smooth as a function of the conserved variables  $u$  and  $v$ . From formula (2.1) it is clear that  $\|f_u\|$  is a Lipschitz constant for the conserved variables  $u$  and  $v$ . It is not hard to check that  $\|f_\gamma\| + \frac{1}{2}L_{u\gamma}$  is a Lipschitz constant for  $h$  with respect to the variable  $\gamma$ . We also recall the decomposition

$$(2.2) \quad \Delta_-^u h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) = \Delta_+^u h_-(\gamma_{j+\frac{1}{2}}, U_j^n) + \Delta_-^u h_+(\gamma_{j+\frac{1}{2}}, U_j^n),$$

where  $h_-, h_+$  are defined in (2.3) via

$$(2.3) \quad h_-(\gamma_{j+\frac{1}{2}}, \xi) = \int_0^\xi f_u^-(\gamma_{j+\frac{1}{2}}, w) dw, \quad h_+(\gamma_{j+\frac{1}{2}}, \xi) = \int_0^\xi f_u^+(\gamma_{j+\frac{1}{2}}, w) dw.$$

It is not hard to check that  $h_-, h_+ \in \text{Lip}([\underline{\gamma}, \bar{\gamma}] \times [0, 1])$  with Lipschitz constants  $\|f_u\|$  and  $L_{u\gamma}$ .

The difference scheme that is analyzed in this paper can then be stated as follows:

$$(2.4) \quad U_j^{n+1} = U_j^n - \lambda \Delta_- h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) + \mu \Delta_- \Delta_+ A(U_j^n),$$

for  $j \in \mathbf{Z}$ ,  $n = 0, \dots, N-1$ , and  $\lambda, \mu$  denoting the numbers

$$\lambda = \frac{\Delta t}{\Delta x}, \quad \mu = \frac{\Delta t}{\Delta x^2} = \frac{\lambda}{\Delta x}.$$

The iteration (2.4) is started by setting

$$(2.5) \quad U_j^0 = \frac{1}{2}(u_0(x_{j-}) + u_0(x_{j+})),$$

and the discretization of  $\gamma$  is *staggered* with respect to that of  $u$ :

$$(2.6) \quad \gamma_{j+\frac{1}{2}} = \frac{1}{2}(\gamma(x_{j+\frac{1}{2}}-) + \gamma(x_{j+\frac{1}{2}}+)).$$

Observe that the definitions of  $U_j^0, \gamma_{j+\frac{1}{2}}$  are meaningful since  $u_0, \gamma \in BV(\mathbf{R})$ .

It can be shown (see Lemma 3.2) that the difference scheme (2.4) is monotone and  $U_j^n \in [0, 1]$  with the CFL condition

$$(2.7) \quad 2\lambda\|f_u\| + 2\mu\|A'\| \leq 1.$$

In the case where  $\|A'\| > 0$ , we will have  $\lambda = \mathcal{O}(\Delta x)$ . In fact, then we can assume that there is a constant  $0 < b \leq 1$  with

$$\lambda = \frac{b\Delta x}{2\|f_u\|\Delta x + 2\|A'\|} \leq \frac{b}{2\|A'\|}\Delta x.$$

In the totally degenerate case (pure convection),  $\|A'\| = 0$ , and then we can allow a less restrictive CFL condition of the form  $\lambda = \mathcal{O}(1)$ . For the remainder of this paper we will assume that  $\|A'\| > 0$ .

Let  $N = \lfloor T/\Delta t \rfloor$ , and  $Z_N^+ = \{1, \dots, N\}$ . The difference solution  $\{U_j^n\}$  constructed via the scheme (2.4) is extended to all of  $\Pi_T$  by defining

$$(2.8) \quad u^\Delta(x, t) = \sum_{n \in Z_N^+} \sum_{j \in \mathbf{Z}} \chi_j^n(x, t) U_j^n, \quad (x, t) \in \Pi_T,$$

where  $\Delta = (\Delta x, \Delta t)$ . Similarly, the discrete coefficient  $\{\gamma_{j+\frac{1}{2}}\}$  is extended to all of  $\mathbf{R}$  by defining

$$\gamma^\Delta(x) = \sum_{j \in \mathbf{Z}} \chi_{j+\frac{1}{2}}(x) \gamma_{j+\frac{1}{2}}, \quad x \in \mathbf{R},$$

where  $\chi_{j+\frac{1}{2}}$  is the characteristic function for the interval  $I_{j+\frac{1}{2}} = [x_j, x_{j+1})$ .

Before proceeding to the proof of the main result of the paper, we provide the following as motivation for the singular mapping approach. In the situation considered in the following lemma, we can actually bound the spatial variation of  $f(U_j^n)$  directly. Note however that we are actually bounding the variation of  $f(U_j^n)$ , and because  $f' > 0$ , this leads to a variation bound on  $U_j^n$ .

**Proposition 2.1.** *In addition to the assumptions on the data described in Section 1, assume that  $f$  is smooth,  $f(u) \rightarrow 0$  as  $u \rightarrow 0$ , and  $f'$  is bounded away from zero,  $f'(u) > f'_{\min} > 0$ . Also assume that  $\underline{\gamma} > 0$ . If the scheme*

$$(2.9) \quad U_j^{n+1} = U_j^n - \lambda \left( \gamma_{j+\frac{1}{2}} f(U_j^n) - \gamma_{j-\frac{1}{2}} f(U_{j-1}^n) \right).$$

*is used to generate approximations  $\{U_j^n\}$  to the conservation law (1.2), and the CFL condition (2.7) is satisfied, the following spatial variation bounds hold uniformly in  $\Delta$ :*

$$(2.10) \quad \sum_{j \in \mathbf{Z}} |\gamma_{j+1/2} f(U_j^n) - \gamma_{j-1/2} f(U_{j-1}^n)| \leq C_1, \quad \sum_{j \in \mathbf{Z}} |U_{j+1}^n - U_j^n| \leq C_2.$$

*Proof.* To prove the first estimate in (2.10),

$$\sum_{j \in \mathbf{Z}} \left| \Delta_- \left( \gamma_{j+1/2} f(U_j^n) \right) \right| = \frac{1}{\lambda} \sum_{j \in \mathbf{Z}} |U_j^{n+1} - U_j^n| \leq C,$$

by Lemma 3.3. For the second estimate,

$$\begin{aligned} \sum_{j \in \mathbf{Z}} |f(U_j^n) - f(U_{j-1}^n)| &\leq \frac{1}{\underline{\gamma}} \sum_{j \in \mathbf{Z}} |\gamma_{j+1/2} f(U_j^n) - \gamma_{j+1/2} f(U_{j-1}^n)| \\ &\leq \frac{1}{\underline{\gamma}} \sum_{j \in \mathbf{Z}} \left( |\gamma_{j+1/2} f(U_j^n) - \gamma_{j-1/2} f(U_{j-1}^n)| + |\gamma_{j-1/2} f(U_{j-1}^n) - \gamma_{j+1/2} f(U_{j-1}^n)| \right) \\ &\leq \frac{1}{\underline{\gamma}} \sum_{j \in \mathbf{Z}} \left| \Delta_- \left( \gamma_{j+1/2} f(U_j^n) \right) \right| + \frac{1}{\underline{\gamma}} \|f\| \sum_{j \in \mathbf{Z}} |\Delta_- \gamma_{j+1/2}|, \end{aligned}$$

and this is uniformly bounded, using the first estimate in (2.10) and the fact that  $\gamma \in BV$ . Now, by the mean value theorem,

$$\sum_{j \in \mathbf{Z}} |f(U_j^n) - f(U_{j-1}^n)| = \sum_{j \in \mathbf{Z}} |f'(\theta_j^n)| |U_j^n - U_{j-1}^n|,$$

for some  $\theta_j^n$  between  $U_j^n$  and  $U_{j-1}^n$ , from which it follows that

$$C \geq \sum_{j \in \mathbf{Z}} |f(U_j^n) - f(U_{j-1}^n)| \geq f'_{\min} \sum_{j \in \mathbf{Z}} |U_j^n - U_{j-1}^n|,$$

completing the proof.  $\square$

**Remark 2.1.** *For example, the flux in Proposition 2.1 could be linear,  $f(u) = u$ .*

For the conservation law (1.2), since  $f' > 0$ , the singular mapping  $\Psi$  defined by (3.1) in the next section reduces to

$$\Psi(\gamma, u) = \gamma f(u),$$

and so the first estimate in (2.10) gives a spatial variation bound for the transformed variable,  $z_j^n = \Psi(\gamma_j, U_j^n)$ . That bound is then used to derive a variation bound on the conserved quantity itself. Clearly, when  $f'$  changes sign the argument proving that  $TV(u) < \infty$  breaks down, and it is possible to construct examples where  $TV(u)$  actually blows up. This variation blow-up can be viewed as resulting from resonance. On the other hand, the bound on  $TV(z)$  remains valid even if  $f'$  vanishes. This leads to the idea of proving compactness for the transformed variable  $z$ , and then using the fact that the singular mapping  $z = \Psi(\gamma, u)$  is strictly increasing as a function of  $u$  to recover the limiting value of the conserved quantity. Of course when  $f'$  changes sign, the version of the singular mapping presented above is not monotone. To see how to circumvent this problem, suppose that the flux  $f(u)$  has a single maximum at  $u^*$ , and notice that the *entropy* flux  $F(u) = \text{sign}(u - u^*)(f(u) - f(u^*))$  is strictly monotone, making it possible to use  $\gamma F(u)$  as the singular mapping, at least in the purely hyperbolic case. When a degenerate diffusion term is present, a somewhat more complicated, but closely related mapping is required, as will be seen in the next section.

### 3. COMPACTNESS OF APPROXIMATE SOLUTIONS

In this section, the goal is to prove strong compactness of our approximate solution  $u^\Delta$  and that any limit of a converging subsequence of  $u^\Delta$  is a weak solution of (1.1).

In what follows we will be studying approximate solutions as the mesh size  $\Delta = (\Delta x, \Delta t)$  decreases. We will always assume that the mesh refinement parameter  $\Delta$  is decreasing with  $\lambda$  constant if  $\|A'\| = 0$  (the purely hyperbolic case), or  $\mu$  constant if  $\|A'\| \neq 0$ , the constant in each case determined by an appropriate CFL condition.

As previously mentioned, our approach is to prove a uniform variation bound with respect to a transformed quantity  $z = \Psi(\gamma, u)$ . The singular mapping  $\Psi(\gamma, u)$  is designed to be Lipschitz continuous and strictly increasing as a function of  $u$ . Due to the presence of the diffusion term  $A(u)$ , it is necessary to modify the singular mapping somewhat from the purely hyperbolic setting, where the singular mapping would simply be

$$\Psi_{\text{hyp}}(\gamma, u) = \int_0^u |f_u(\gamma, w)| dw.$$

We add the diffusion term, and at the same time, zero out the contribution of the convective flux wherever  $A(u)$  is nondegenerate. This allows us to analyze the convective portion  $\mathcal{F}(\gamma, u)$  and the diffusive portion  $A(u)$  separately. Let  $\mathcal{S}$  be the characteristic function for  $\bigcup_i [\alpha_i, \beta_i]$ . The singular mapping is then

$$(3.1) \quad \Psi(\gamma, u) = \int_0^u \mathcal{S}(w) |f_u(\gamma, w)| dw + A(u) =: \mathcal{F}(\gamma, u) + A(u).$$

**Lemma 3.1.** *Under the assumptions described in Section 1, the mapping  $\Psi(\gamma, u)$  is strictly increasing as a function of  $u$ . Furthermore, both  $\Psi$  and  $\mathcal{F}$  belong to  $\text{Lip}([\underline{\gamma}, \bar{\gamma}] \times [0, 1])$ .*

*Proof.* For Lipschitz continuity of  $\Psi$  with respect to  $u$ ,

$$\begin{aligned} |\Psi(\gamma, u) - \Psi(\gamma, v)| &\leq \left| \int_v^u |f_u(\gamma, w)| dw \right| + |A(u) - A(v)| \\ &\leq (\|f_u\| + \|A'\|) |u - v|, \end{aligned}$$

and for Lipschitz continuity as a function of  $\gamma$ ,

$$\begin{aligned} |\Psi(\gamma_1, u) - \Psi(\gamma_2, u)| &= \left| \int_0^u \mathcal{S}(w) |f_u(\gamma_1, w)| dw - \int_0^u \mathcal{S}(w) |f_u(\gamma_2, w)| dw \right| \\ &\leq \left| \int_0^u |f_u(\gamma_1, w) - f_u(\gamma_2, w)| dw \right| \\ &\leq |u - 0| L_{u\gamma} |\gamma_1 - \gamma_2| \leq L_{u\gamma} |\gamma_1 - \gamma_2|. \end{aligned}$$



It is clear that essentially the same estimates prove that also  $\mathcal{F} \in \text{Lip}([\underline{\gamma}, \bar{\gamma}] \times [0, 1])$ . By definition,

$$\Psi_u(\gamma, u) = \mathcal{S}(u)|f_u(\gamma, u)| + (1 - \mathcal{S}(u))A'(u) > 0 \quad \text{a.e.},$$

which proves strict monotonicity.  $\square$

We will use the notation  $\|\Psi_u\|$ ,  $\|\Psi_\gamma\|$ ,  $\|\mathcal{F}_u\|$ , and  $\|\mathcal{F}_\gamma\|$  for the Lipschitz constants provided by the previous lemma. In what follows,  $C_i$  will denote positive constants that can depend on the data of the problem but not on  $\Delta$ .

The approach will be to show compactness for the sequence of transformed functions

$$z^\Delta(x, t) = \Psi(\gamma^\Delta(x), u^\Delta(x, t)),$$

where  $u^\Delta(x, t)$  denotes the numerical approximation generated by the scheme.

A finite difference scheme such as the scheme (2.4) is monotone [10, 22] if

$$(3.2) \quad U_j^n \leq V_j^n \quad \forall j \implies U_j^{n+1} \leq V_j^{n+1} \quad \forall j.$$

**Lemma 3.2.** *If the assumptions concerning the data described in Section 1 are satisfied, and the scheme (2.4) is applied with the parameters  $\lambda$  and  $\mu$  chosen so that the following CFL condition is satisfied*

$$(3.3) \quad 2\lambda\|f_u\| + 2\mu\|A'\| \leq 1.$$

*Then the computed solutions remain in the interval  $[0, 1]$ , the CFL condition (3.3) holds for each succeeding time step, and the scheme (2.4) is monotone.*

*Proof.* The formula (2.4) defines  $U_j^{n+1}$  as a function

$$U_j^{n+1} = G_j(U_{j+1}^n, U_j^n, U_{j-1}^n, \gamma_{j+\frac{1}{2}}, \gamma_{j-\frac{1}{2}}).$$

The partial derivatives with respect to the conserved variables are

$$\begin{aligned} \partial U_j^{n+1} / \partial U_{j+1}^n &= -\lambda f_u^-(\gamma_{j+\frac{1}{2}}, U_{j+1}^n) + \mu A'(U_{j+1}^n) \geq 0, \\ \partial U_j^{n+1} / \partial U_{j-1}^n &= \lambda f_u^+(\gamma_{j-\frac{1}{2}}, U_{j-1}^n) + \mu A'(U_{j-1}^n) \geq 0, \\ \partial U_j^{n+1} / \partial U_j^n &= 1 + \lambda f_u^-(\gamma_{j+\frac{1}{2}}, U_j^n) - \lambda f_u^+(\gamma_{j-\frac{1}{2}}, U_j^n) - 2\mu A'(U_j^n). \end{aligned}$$

Thus  $U_j^{n+1}$  is a nondecreasing function of the conserved variables at the lower time level if

$$1 + \lambda f_u^-(\gamma_{j+\frac{1}{2}}, U_j^n) - \lambda f_u^+(\gamma_{j-\frac{1}{2}}, U_j^n) - 2\mu A'(U_j^n) \geq 0.$$

This will hold if the CFL condition (3.3) is satisfied for the solution at level  $n$ . If the CFL condition holds for the initial data, then since each of the functions  $G_j(\cdot, \cdot, \cdot, \gamma_{j+\frac{1}{2}}, \gamma_{j-\frac{1}{2}})$  is nondecreasing as a function of its first three arguments, and  $U_j^0 \in [0, 1]$

$$\begin{aligned} 0 = G_j(0, 0, 0, \gamma_{j+\frac{1}{2}}, \gamma_{j-\frac{1}{2}}) &\leq G_j(U_{j+1}^0, U_j^0, U_{j-1}^0, \gamma_{j+\frac{1}{2}}, \gamma_{j-\frac{1}{2}}) \\ &= U_j^{n+1} \leq G_j(1, 1, 1, \gamma_{j+\frac{1}{2}}, \gamma_{j-\frac{1}{2}}) = 1. \end{aligned}$$

(3.4)

The first and last equalities in this relationship result from the fact that for all  $\gamma$ ,  $f(\gamma, 0) = f_0$  and  $f(\gamma, 1) = f_1$ . Proceeding inductively, it is clear that the solution  $U_j^n \in [0, 1]$  for each  $n \geq 0$ , and thus the CFL condition remains satisfied at each succeeding time level. That the scheme is monotone is clear from the fact that  $U_j^{n+1}$  is a nondecreasing function of  $U_{j+1}^n$ ,  $U_j^n$ , and  $U_{j-1}^n$ .  $\square$

The next lemma is of fundamental importance for the subsequent analysis. In addition to providing for  $L^1$  time continuity of the numerical approximations, it also plays a key role in our bound on the space translates of the transformed variable. With respect to the spatial variation, the situation is somewhat different here than in the case where  $\gamma$  is constant. Specifically, the variation (measured via the transformed variable) may actually increase from one time step to the next, and so the now classical total variation decreasing (TVD) argument is not available. Instead, we bound the spatial variation in terms of the  $L^1$  time translates.

**Lemma 3.3.** *Assume that the hypotheses of Lemma 3.2 are satisfied. Then there exists a constant  $C_3$ , independent of  $\Delta$ , such that*

$$\Delta x \sum_{j \in \mathbf{Z}} |U_j^{n+1} - U_j^n| \leq \Delta x \sum_{j \in \mathbf{Z}} |U_j^1 - U_j^0| \leq C_3 \Delta t.$$

*Proof.* Starting from the marching formula (2.4), the time differences can be expressed as follows:

$$\begin{aligned} U_j^{n+1} - U_j^n &= U_j^n - U_j^{n-1} - \lambda \Delta_- \left( h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) - h(\gamma_{j+\frac{1}{2}}, U_{j+1}^{n-1}, U_j^{n-1}) \right) \\ &\quad + \mu \Delta_- \Delta_+ (A(U_j^n) - A(U_j^{n-1})) \\ &= \left( 1 - \lambda C_{j+\frac{1}{2}}^{n-\frac{1}{2}} + \lambda B_{j-\frac{1}{2}}^{n-\frac{1}{2}} - 2\mu D_j^{n-\frac{1}{2}} \right) (U_j^n - U_j^{n-1}) \\ &\quad + \left( -\lambda B_{j+\frac{1}{2}}^{n-\frac{1}{2}} + \mu D_{j+1}^{n-\frac{1}{2}} \right) (U_{j+1}^n - U_{j+1}^{n-1}) \\ &\quad + \left( \lambda C_{j-\frac{1}{2}}^{n-\frac{1}{2}} + \mu D_{j-1}^{n-\frac{1}{2}} \right) (U_{j-1}^n - U_{j-1}^{n-1}), \end{aligned}$$

where

$$\begin{aligned} B_{j+\frac{1}{2}}^{n-\frac{1}{2}} &= \int_0^1 f_u^-(\gamma_{j+\frac{1}{2}}, \theta U_{j+1}^n + (1-\theta)U_{j+1}^{n-1}) d\theta \leq 0, \\ C_{j+\frac{1}{2}}^{n-\frac{1}{2}} &= \int_0^1 f_u^+(\gamma_{j+\frac{1}{2}}, \theta U_j^n + (1-\theta)U_j^{n-1}) d\theta \geq 0, \\ D_j^{n-\frac{1}{2}} &= \frac{A(U_j^n) - A(U_j^{n-1})}{U_j^n - U_j^{n-1}} \geq 0. \end{aligned}$$

Due to the CFL condition (3.3),

$$(3.5) \quad 1 - \lambda C_{j+\frac{1}{2}}^{n-\frac{1}{2}} + \lambda B_{j-\frac{1}{2}}^{n-\frac{1}{2}} - 2\mu D_j^{n-\frac{1}{2}} \geq 0,$$

and so

$$\begin{aligned} |U_j^{n+1} - U_j^n| &\leq \left( 1 - \lambda C_{j+\frac{1}{2}}^{n-\frac{1}{2}} + \lambda B_{j-\frac{1}{2}}^{n-\frac{1}{2}} - 2\mu D_j^{n-\frac{1}{2}} \right) |U_j^n - U_j^{n-1}| \\ (3.6) \quad &\quad + \left( -\lambda B_{j+\frac{1}{2}}^{n-\frac{1}{2}} + \mu D_{j+1}^{n-\frac{1}{2}} \right) |U_{j+1}^n - U_{j+1}^{n-1}| \\ &\quad + \left( \lambda C_{j-\frac{1}{2}}^{n-\frac{1}{2}} + \mu D_{j-1}^{n-\frac{1}{2}} \right) |U_{j-1}^n - U_{j-1}^{n-1}|. \end{aligned}$$

Summing this inequality over  $j$  and multiplying by  $\Delta x$  gives

$$\Delta x \sum_{j \in \mathbf{Z}} |U_j^{n+1} - U_j^n| \leq \Delta x \sum_{j \in \mathbf{Z}} |U_j^n - U_j^{n-1}|.$$

Continuing this way by induction yields

$$\Delta x \sum_{j \in \mathbf{Z}} |U_j^{n+1} - U_j^n| \leq \Delta x \sum_{j \in \mathbf{Z}} |U_j^1 - U_j^0|.$$

Then, using Lipschitz continuity of  $h$  and the fact that  $\lambda \Delta x = \Delta t$ ,

$$\begin{aligned} \Delta x \sum_{j \in \mathbf{Z}} |U_j^1 - U_j^0| &= \Delta t \sum_{j \in \mathbf{Z}} \left| \Delta_- h(\gamma_{j+\frac{1}{2}}, U_{j+1}^0, U_j^0) - \Delta_- \frac{1}{\Delta x} \Delta_+ A(U_j^0) \right| \\ &\leq \Delta t \sum_{j \in \mathbf{Z}} |\Delta_- h(\gamma_{j+\frac{1}{2}}, U_{j+1}^0, U_j^0)| + \Delta t \sum_{j \in \mathbf{Z}} \left| \Delta_- \frac{1}{\Delta x} \Delta_+ A(U_j^0) \right| \\ &\leq 2\Delta t \|f_u\| TV(u_0) + \Delta t \left( \|f_\gamma\| + \frac{1}{2} L_{u\gamma} \right) TV(\gamma) + \Delta t \sum_{j \in \mathbf{Z}} \left| \Delta_- \frac{1}{\Delta x} \Delta_+ A(U_j^0) \right|. \end{aligned}$$

We still have to estimate the last sum in the preceding inequality. To this end, notice first that if  $u_0$  has a jump at  $x = x_k$  for some index  $k$ . Then by (1.6),

$$A(U_k^0) = A(u_0(x_k-)) = A(u_0(x_k+)).$$

Now using this, it is clear that

$$\begin{aligned} \left| \Delta_- \frac{1}{\Delta x} \Delta_+ A(U_j^0) \right| &= \left| \Delta_- \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} A(u_0(x))_x dx \right| \\ &= \left| \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} (A(u_0(x))_x - A(u_0(x - \Delta x))_x) dx \right| \\ &\leq \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} |A(u_0(x))_x - A(u_0(x - \Delta x))_x| dx, \end{aligned}$$

and so summing over  $j$  and using standard results concerning total variation (e.g., Lemma 2.4.4 of [27]) yields

$$\begin{aligned} \sum_{j \in \mathbf{Z}} \left| \Delta_- \frac{1}{\Delta x} \Delta_+ A(U_j^0) \right| &\leq \frac{1}{\Delta x} \int_{\mathbf{R}} |A(u_0(x))_x - A(u_0(x - \Delta x))_x| dx \\ &\leq TV(A(u_0)_x), \end{aligned}$$

which gives the desired bound and completes the proof.  $\square$

**Lemma 3.4.** *Under the assumptions in Lemma 3.2, the computed solutions  $u^\Delta(\cdot, t^n)$  satisfy a uniform  $L^1(\mathbf{R})$  bound for  $t^n \in [0, T]$ :*

$$(3.7) \quad \|u^\Delta(\cdot, t^n)\|_{L^1(\mathbf{R})} \leq C_3 T + \|u_0\|_{L^1(\mathbf{R})} + \frac{1}{2} \Delta x TV(u_0),$$

and if  $v^\Delta$  is another solution generated using the same discretization of  $\Pi_T$ , the following discrete  $L^1$  contraction property holds:

$$(3.8) \quad \|u^\Delta(\cdot, t^n) - v^\Delta(\cdot, t^n)\|_{L^1(\mathbf{R})} \leq \|u^\Delta(\cdot, 0) - v^\Delta(\cdot, 0)\|_{L^1(\mathbf{R})}.$$

*Proof.* Using the triangle inequality and the result of Lemma 3.3 yields

$$\begin{aligned} \|u^\Delta(\cdot, t^n)\|_{L^1(\mathbf{R})} &= \Delta x \sum_{j \in \mathbf{Z}} |U_j^n| \leq \Delta x \sum_{j \in \mathbf{Z}} |U_j^n - U_j^{n-1}| + \Delta x \sum_{j \in \mathbf{Z}} |U_j^{n-1}| \\ &\leq \Delta x \sum_{j \in \mathbf{Z}} |U_j^1 - U_j^0| + \Delta x \sum_{j \in \mathbf{Z}} |U_j^{n-1}|. \end{aligned}$$

Proceeding by induction,

$$\|u^\Delta(\cdot, t^n)\|_{L^1(\mathbf{R})} \leq n \Delta x \sum_{j \in \mathbf{Z}} |U_j^1 - U_j^0| + \Delta x \sum_{j \in \mathbf{Z}} |U_j^0|.$$

By Lemma 3.3,  $n \Delta x \sum_{j \in \mathbf{Z}} |U_j^1 - U_j^0| \leq n C_3 \Delta t \leq C_3 T$ . Now applying the estimate

$$\begin{aligned} \Delta x \sum_{j \in \mathbf{Z}} |U_j^0| &\leq \int_{\mathbf{R}} |u_0(x)| dx + \sum_{j \in \mathbf{Z}} \int_{I_j} |u_0(x) - U_j^0| dx \\ &\leq \int_{\mathbf{R}} |u_0(x)| dx + \frac{1}{2} \Delta x TV(u_0), \end{aligned}$$

the proof of (3.7) is complete. The discrete  $L^1$  contraction property (3.8) follows from the Crandall-Tartar lemma [10], using the fact that the operator which advances the initial approximation to time level  $n$  is monotone, conservative, and takes  $L^1$  mesh functions into  $L^1$  mesh functions.  $\square$

The next three lemmas provide a proof of compactness for the sequence of functions  $\mathcal{F}^\Delta$  defined by

$$\mathcal{F}^\Delta(x, t) = \mathcal{F}(\gamma^\Delta(x), u^\Delta(x, t)).$$

In what follows, we will use the Kruřkov entropy-entropy flux pair indexed by  $c$ :

$$V(u) = |u - c|, \quad F(\gamma, u) = \sigma(u - c)(f(\gamma, u) - f(\gamma, c)).$$

**Lemma 3.5.** *Assume that the hypotheses of Lemma 3.2 are satisfied. For each  $c \in \mathbf{R}$ ,*

$$(3.9) \quad V(U_j^{n+1}) \leq V(U_j^n) - \lambda \Delta_-^u H(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) + \mu \Delta_- \Delta_+ |A(U_j^n) - A(c)| + \lambda C_4 |\Delta_+ \gamma_{j-\frac{1}{2}}|,$$

where the EO numerical entropy flux is given by

$$(3.10) \quad H(\gamma, v, u) = \frac{1}{2}(F(\gamma, u) + F(\gamma, v)) - \frac{1}{2} \int_u^v \sigma(w - c) |f_u(\gamma, w)| dw.$$

*Proof.* Let  $a \vee b = \max(a, b)$  and  $a \wedge b = \min(a, b)$ . With

$$\rho_j^{n+1} = U_j^n - \lambda \Delta_-^u h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) + \mu \Delta_- \Delta_+ A(U_j^n),$$

the following discrete entropy inequality follows from Lemma 3.7 of [18]:

$$V(\rho_j^{n+1}) \leq V(U_j^n) - \lambda \Delta_-^u H(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) + \mu \Delta_- \Delta_+ |A(U_j^n) - A(c)|,$$

where

$$(3.11) \quad H(\gamma, v, u) = h(\gamma, v \vee c, u \vee c) - h(\gamma, v \wedge c, u \wedge c).$$

For a derivation of the explicit formula (3.10) for the EO numerical entropy flux from (3.11), see [45]. Then

$$\begin{aligned} V(U_j^{n+1}) &\leq V(U_j^n) - \lambda \Delta_-^u H(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) + \mu \Delta_- \Delta_+ |A(U_j^n) - A(c)| \\ &\quad - V(\rho_j^{n+1}) + V(U_j^{n+1}). \end{aligned}$$

It remains to estimate  $V(\rho_j^{n+1}) - V(U_j^{n+1})$ :

$$\begin{aligned} |V(\rho_j^{n+1}) - V(U_j^{n+1})| &\leq |\rho_j^{n+1} - U_j^{n+1}| \\ &= \lambda |\Delta_- h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) - \Delta_-^u h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n)| \\ &\leq \lambda \left( \|f_\gamma\| + \frac{1}{2} L_{u\gamma} \right) |\gamma_{j+\frac{1}{2}} - \gamma_{j-\frac{1}{2}}| \\ (3.12) \quad &= \lambda C_4 |\Delta_+ \gamma_{j-\frac{1}{2}}|. \end{aligned}$$

□

In what follows,  $\chi_r(w; c)$  is the characteristic function for the interval  $[c, +\infty)$ ,  $\chi_l(w; c)$  is the characteristic function for  $(-\infty, c]$ , and  $\chi_{(\alpha, \beta]}(w)$  is the characteristic function of the interval  $(\alpha, \beta]$ . The following identities (see [45] for a derivation) will be required in the proof of Lemma 3.6:

$$\begin{aligned} (3.13) \quad &\frac{1}{2} \left( \Delta_-^u H(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) + \Delta_-^u h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) \right) \\ &= \int_{U_j^n}^{U_{j+1}^n} \chi_r(w; c) f_u^-(\gamma_{j+\frac{1}{2}}, w) dw + \int_{U_{j-1}^n}^{U_j^n} \chi_r(w; c) f_u^+(\gamma_{j+\frac{1}{2}}, w) dw, \end{aligned}$$

$$\begin{aligned} (3.14) \quad &\frac{1}{2} \left( \Delta_-^u H(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) - \Delta_-^u h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) \right) \\ &= - \int_{U_j^n}^{U_{j+1}^n} \chi_l(w; c) f_u^-(\gamma_{j+\frac{1}{2}}, w) dw - \int_{U_{j-1}^n}^{U_j^n} \chi_l(w; c) f_u^+(\gamma_{j+\frac{1}{2}}, w) dw. \end{aligned}$$

**Lemma 3.6.** *Assume that the hypotheses of Lemma 3.2 are satisfied. Let  $u^*(\gamma)$  denote the unique maximum of  $f(\gamma, u)$  for  $u \in [0, 1]$ . The following inequality holds for  $c \geq u^*(\gamma_{j+\frac{1}{2}})$ :*

$$\begin{aligned} (3.15) \quad &-\Delta_-^u \left( \int_0^{U_{j+1}^n} \chi_r(w; c) |f_u(\gamma_{j+\frac{1}{2}}, w)| dw + \frac{1}{\Delta x} \Delta_+ (A(U_j^n) - A(c))_+ \right) \\ &\leq \frac{1}{\lambda} |U_j^{n+1} - U_j^n| + C_5 |\Delta_+ \gamma_{j-\frac{1}{2}}|, \end{aligned}$$

and the following inequality holds for  $c \leq u^*(\gamma_{j-\frac{1}{2}})$ :

$$(3.16) \quad -\Delta_-^u \left( \int_0^{U_j^n} \chi_l(w; c) |f_u(\gamma_{j-\frac{1}{2}}, w)| dw - \frac{1}{\Delta x} \Delta_+ (A(U_j^n) - A(c))_- \right) \leq \frac{1}{\lambda} |U_j^{n+1} - U_j^n| + C_5 |\Delta_+ \gamma_{j-\frac{1}{2}}|.$$

*Proof.* We start by writing the scheme (2.4) as

$$(3.17) \quad \begin{aligned} \Delta_-^u h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) - \frac{1}{\Delta x} \Delta_- \Delta_+ (A(U_j^n) - A(c)) \\ = \frac{1}{\lambda} (U_j^n - U_j^{n+1}) + (\Delta_-^u - \Delta_-) h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n). \end{aligned}$$

Next, we write the discrete entropy inequality (3.9) as

$$(3.18) \quad \begin{aligned} \Delta_-^u H(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) - \frac{1}{\Delta x} \Delta_- \Delta_+ |A(U_j^n) - A(c)| \\ \leq \frac{1}{\lambda} (V(U_j^n) - V(U_j^{n+1})) + C_4 |\Delta_+ \gamma_{j-\frac{1}{2}}| \\ \leq \frac{1}{\lambda} |U_j^n - U_j^{n+1}| + C_4 |\Delta_+ \gamma_{j-\frac{1}{2}}|. \end{aligned}$$

Inequality (3.15) is derived by adding (3.17) and (3.18), and then dividing by two. The right hand side of this combination is

$$\begin{aligned} \frac{1}{2} \left( \frac{1}{\lambda} (U_j^n - U_j^{n+1}) + (\Delta_-^u - \Delta_-) h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) + \frac{1}{\lambda} |U_j^n - U_j^{n+1}| + C_4 |\Delta_+ \gamma_{j-\frac{1}{2}}| \right) \\ = \frac{1}{\lambda} (U_j^n - U_j^{n+1})_+ + \frac{1}{2} (\Delta_-^u - \Delta_-) h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) + \frac{1}{2} C_4 |\Delta_+ \gamma_{j-\frac{1}{2}}| \\ \leq \frac{1}{\lambda} |U_j^n - U_j^{n+1}| + C_5 |\Delta_+ \gamma_{j-\frac{1}{2}}|, \end{aligned}$$

where we have used the fact that the numerical flux  $h$  is Lipschitz continuous as a function of  $\gamma$ . Applying (3.13), the left side of the combination is

$$\begin{aligned} \frac{1}{2} \left( \Delta_-^u h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) + \Delta_-^u H(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) \right) \\ - \frac{1}{2} \left( \frac{1}{\Delta x} \Delta_- \Delta_+ (A(U_j^n) - A(c)) + \frac{1}{\Delta x} \Delta_- \Delta_+ |A(U_j^n) - A(c)| \right) \\ = \int_{U_j^n}^{U_{j+1}^n} \chi_r(w; c) f_u^-(\gamma_{j+\frac{1}{2}}, w) dw + \int_{U_{j-1}^n}^{U_j^n} \chi_r(w; c) f_u^+(\gamma_{j+\frac{1}{2}}, w) dw \\ - \frac{1}{\Delta x} \Delta_- \Delta_+ (A(U_j^n) - A(c))_+. \end{aligned}$$

With  $c \geq u^*(\gamma_{j+\frac{1}{2}})$ , the integral involving  $\chi_r(w; c) f_u^+(\gamma_{j+\frac{1}{2}}, w)$  is zero, and

$$\chi_r(w; c) f_u^-(\gamma_{j+\frac{1}{2}}, w) = -\chi_r(w; c) |f_u(\gamma_{j+\frac{1}{2}}, w)|,$$

completing the proof of (3.15). Inequality (3.16) is proven in a similar way by subtracting (3.17) from (3.18) and dividing by two, and then using (3.14).  $\square$

Let

$$(3.19) \quad c^R(\gamma) = u^*(\gamma) \vee c, \quad c^L(\gamma) = u^*(\gamma) \wedge c$$

and

$$(3.20) \quad \psi^R(\gamma, c, u) = \int_0^u \chi_r(w; c^R(\gamma)) |f_u(\gamma, w)| dw + \frac{1}{\Delta x} \Delta_+ (A(u) - A(c^R(\gamma)))_+,$$

$$(3.21) \quad \psi^L(\gamma, c, u) = \int_0^u \chi_l(w; c^L(\gamma)) |f_u(\gamma, w)| dw - \frac{1}{\Delta x} \Delta_+ (A(u) - A(c^L(\gamma)))_-.$$

**Lemma 3.7.** *Assume that the hypotheses of Lemma 3.2 are satisfied. For some constant  $C_6$ , independent of the mesh size  $\Delta$ ,*

$$\sum_{j \in \mathbf{Z}} \left| \Delta_-^u \mathcal{F}(\gamma_{j-\frac{1}{2}}, U_j^n) \right| \leq C_6.$$

*Proof.* The first step of the proof is to establish a uniform (with respect to  $\Delta$ ) bound for the sequence of divided differences  $\Delta_+ A(U_j^n)/\Delta x$ . We postpone the proof of this result to Lemma 4.1. In addition, it is clear that for any constant  $\kappa$ , the same uniform bound also holds for the divided differences  $\Delta_+(A(U_j^n) - \kappa)_-/\Delta x$  and  $\Delta_+(A(U_j^n) - \kappa)_+/\Delta x$ .

To continue the proof, fix  $i \in \{1, \dots, M\}$ , and take  $c = \alpha_i$ ,  $c = \beta_i$  in the formulas (3.19), (3.20), (3.21). Using the fact that  $A$  is constant on the interval  $[\alpha_i, \beta_i]$ , we get

$$\begin{aligned} \psi^L(\gamma, \beta_i, u) - \psi^L(\gamma, \alpha_i, u) &= \int_0^u \chi_{(\alpha_i^L(\gamma), \beta_i^L(\gamma))}(w) |f_u(\gamma, w)| dw, \\ \psi^R(\gamma, \alpha_i, u) - \psi^R(\gamma, \beta_i, u) &= \int_0^u \chi_{(\alpha_i^R(\gamma), \beta_i^R(\gamma))}(w) |f_u(\gamma, w)| dw, \end{aligned}$$

where

$$\alpha_i^R = u^*(\gamma) \vee \alpha_i, \quad \beta_i^R = u^*(\gamma) \vee \beta_i, \quad \alpha_i^L = u^*(\gamma) \wedge \alpha_i, \quad \beta_i^L = u^*(\gamma) \wedge \beta_i.$$

As a consequence, with the help of the identity

$$\chi_{(\alpha_i^L(\gamma), \beta_i^L(\gamma))} + \chi_{(\alpha_i^R(\gamma), \beta_i^R(\gamma))} = \chi_{(\alpha_i, \beta_i)},$$

we find that

$$\begin{aligned} \mathcal{F}_i(\gamma, u) &:= \int_0^u \chi_{(\alpha_i, \beta_i)}(w) |f_u(\gamma, w)| dw \\ &= \int_0^u \left( \chi_{(\alpha_i^L(\gamma), \beta_i^L(\gamma))}(w) + \chi_{(\alpha_i^R(\gamma), \beta_i^R(\gamma))}(w) \right) |f_u(\gamma, w)| dw \\ &= \psi^L(\gamma, \beta_i, u) - \psi^L(\gamma, \alpha_i, u) + \psi^R(\gamma, \alpha_i, u) - \psi^R(\gamma, \beta_i, u). \end{aligned}$$

Now observing that  $\mathcal{F}$  has the decomposition

$$(3.22) \quad \mathcal{F}(\gamma, u) = \sum_{i=1}^M \mathcal{F}_i(\gamma, u),$$

it suffices to show that the sum

$$\sum_{j \in \mathbf{Z}} |\Delta_-^u \mathcal{F}_i(\gamma_{j-\frac{1}{2}}, U_j^n)|$$

is bounded independently of the mesh size  $\Delta$ , for each  $i = 1, \dots, M$ .

Recalling Lemma 3.6, for each  $c \in \mathbf{R}$  we have

$$(3.23) \quad -\Delta_-^u \psi^L(\gamma_{j-\frac{1}{2}}, c, U_j^n) \leq \frac{1}{\lambda} |U_j^{n+1} - U_j^n| + C_5 |\Delta_+ \gamma_{j-\frac{1}{2}}|,$$

$$(3.24) \quad -\Delta_-^u \psi^R(\gamma_{j-\frac{1}{2}}, c, U_j^n) \leq \frac{1}{\lambda} |U_{j-1}^{n+1} - U_{j-1}^n| + C_5 |\Delta_- \gamma_{j-\frac{1}{2}}|.$$

If the quantities  $A(c^L(\gamma))$ ,  $A(c^R(\gamma))$  appearing in (3.20), (3.21) are independent of  $\gamma$ , then both  $\psi^L(\gamma, c, u)$  and  $\psi^R(\gamma, c, u)$  are Lipschitz continuous in  $\gamma$ ; this is a result of the Lipschitz continuity (with respect to  $\gamma$ ) of  $f_u$  and of  $u^*$ . When that is the case, by replacing  $C_5$  in (3.23), (3.24) by a larger constant  $C_7$ , we can replace the  $\Delta_-^u$  difference operators with  $\Delta_-$  operators:

$$(3.25) \quad -\Delta_- \psi^L(\gamma_{j-\frac{1}{2}}, c, U_j^n) \leq \frac{1}{\lambda} |U_j^{n+1} - U_j^n| + C_7 |\Delta_+ \gamma_{j-\frac{1}{2}}|,$$

$$(3.26) \quad -\Delta_- \psi^R(\gamma_{j-\frac{1}{2}}, c, U_j^n) \leq \frac{1}{\lambda} |U_{j-1}^{n+1} - U_{j-1}^n| + C_7 |\Delta_- \gamma_{j-\frac{1}{2}}|.$$

Since the right hand sides of (3.25) and (3.26) are nonnegative, we can replace the left sides of (3.25) and (3.26) by the nonnegative quantities  $-(\Delta_- \psi^L(\gamma_{j-\frac{1}{2}}, c, U_j^n))_-$  and  $-(\Delta_- \psi^R(\gamma_{j+\frac{1}{2}}, c, U_j^n))_-$ . Then summing over  $j \in \mathbf{Z}$ , we see that both  $\psi^L(\gamma_{j-\frac{1}{2}}, c, U_j^n)$  and  $\psi^R(\gamma_{j-\frac{1}{2}}, c, U_j^n)$  have uniformly

bounded negative variation, as a result of Lemma 3.3. Moreover, still assuming that  $A(c^R)$  and  $A(c^L)$  are independent of  $\gamma$ , both  $\psi^L(\gamma_{j-\frac{1}{2}}, c, U_j^n)$  and  $\psi^R(\gamma_{j-\frac{1}{2}}, c, U_j^n)$  are bounded uniformly in  $\Delta$ . This is a consequence of our bound on the divided differences  $\Delta_+(A(U_j^n) - \kappa)_-/\Delta x$  and  $\Delta_+(A(U_j^n) - \kappa)_+/\Delta x$  (with  $\kappa = A(c^R)$ ,  $\kappa = A(c^L)$ ), along with the fact that  $U_j^n \in [0, 1]$ . Taken together, the bounds on the negative variation and the amplitude allow us to conclude that the total variations are also uniformly bounded:

$$\sum_{j \in \mathbf{Z}} \left| \Delta_- \psi^L(\gamma_{j-\frac{1}{2}}, c, U_j^n) \right| \leq C_8, \quad \sum_{j \in \mathbf{Z}} \left| \Delta_- \psi^R(\gamma_{j-\frac{1}{2}}, c, U_j^n) \right| \leq C_8.$$

Now using Lipschitz continuity (with respect to  $\gamma$ ) of  $\psi^R(\gamma, c, u)$  and  $\psi^R(\gamma, c, u)$  one more time to go back to  $\Delta_-^u$  difference operators, we have

$$\sum_{j \in \mathbf{Z}} \left| \Delta_-^u \psi^L(\gamma_{j-\frac{1}{2}}, c, U_j^n) \right| \leq C_9, \quad \sum_{j \in \mathbf{Z}} \left| \Delta_-^u \psi^R(\gamma_{j-\frac{1}{2}}, c, U_j^n) \right| \leq C_9.$$

If we can apply these estimates to the  $\psi^L$  and  $\psi^R$  terms making up  $\mathcal{F}_i$ , then in view of the decomposition (3.22), an application of the triangle inequality will complete the proof. Thus the remainder of the proof consists of checking that none of the quantities  $A(\alpha_i^L)$ ,  $A(\alpha_i^R)$ ,  $A(\beta_i^L)$ ,  $A(\beta_i^R)$  appearing in  $\mathcal{F}_i$  depend on  $\gamma$ . First take the case where  $u^*(\gamma) \leq \alpha_i \leq \beta_i$ . Because of our assumptions about  $u^*$ , if this relationship holds for any value of  $\gamma$ , it holds for all  $\gamma$ . In this case

$$\psi^L(\gamma, \beta_i, u) = \psi^L(\gamma, \alpha_i, u) = \psi^L(\gamma, u^*(\gamma), u),$$

and thus the  $\psi^L$  terms cancel, leaving

$$\mathcal{F}_i(\gamma, u) = \psi^R(\gamma, \alpha_i, u) - \psi^R(\gamma, \beta_i, u).$$

So, for this case we only have to verify that  $A(\alpha_i^R)$  and  $A(\beta_i^R)$  are independent of  $\gamma$ . With  $u^*(\gamma) \leq \alpha_i \leq \beta_i$ ,  $\alpha_i^R = \alpha_i$  and  $\beta_i^R = \beta_i$ , i.e.,  $\alpha_i^R$  and  $\beta_i^R$  are independent of  $\gamma$ , and thus so are  $A(\alpha_i^R)$ ,  $A(\beta_i^R)$ . Similarly, in the case where  $\alpha_i \leq \beta_i \leq u^*$ ,

$$\mathcal{F}_i(\gamma, u) = \psi^L(\gamma, \beta_i, u) - \psi^L(\gamma, \alpha_i, u),$$

and  $\alpha_i^L = \alpha_i$ ,  $\beta_i^L = \beta_i$ , independently of  $\gamma$ . The last case is the one where  $\alpha_i \leq u^*(\gamma) \leq \beta_i$ . Here, all four terms making up  $\mathcal{F}_i$  are present:

$$\mathcal{F}_i(\gamma, u) = \psi^L(\gamma, \beta_i, u) - \psi^L(\gamma, \alpha_i, u) + \psi^R(\gamma, \alpha_i, u) - \psi^R(\gamma, \beta_i, u),$$

but since  $A$  is constant on  $[\alpha_i, \beta_i]$ ,

$$A(\alpha_i^L) = A(\alpha_i^R) = A(\beta_i^L) = A(\beta_i^R) = A(\alpha_i) = A(\beta_i) = A(u^*(\gamma)),$$

all of which are independent of  $\gamma$ . □

**Lemma 3.8.** *Assume that the hypotheses of Lemma 3.2 are satisfied. There exists a subsequence of  $\{\mathcal{F}^\Delta\}$ , also denoted by  $\{\mathcal{F}^\Delta\}$ , and a function*

$$\overline{\mathcal{F}} \in L^1(\Pi_T) \bigcap L^\infty(\Pi_T)$$

*such that  $\mathcal{F}^\Delta \rightarrow \overline{\mathcal{F}}$  in  $L^1_{\text{loc}}(\Pi_T)$  and a.e. in  $\Pi_T$ . Furthermore,  $\overline{\mathcal{F}}(\cdot, t) \in L^1(\mathbf{R})$  for all  $t \in [0, T]$ .*

*Proof.* The first step of the proof is to establish a uniform variation bound for  $\mathcal{F}^\Delta(\cdot, t^n)$ . The step function  $\mathcal{F}^\Delta(\cdot, t^n)$  has jumps at cell centers, due to jumps in  $\gamma^\Delta$ , plus jumps at cell boundaries, due to jumps in  $u^\Delta$ . When both types of jumps are accounted for, we arrive at

$$\begin{aligned} TV(\mathcal{F}^\Delta(\cdot, t^n)) &= \sum_{j \in \mathbf{Z}} \left| \Delta_-^u \mathcal{F}(\gamma_{j-\frac{1}{2}}, U_j^n) \right| + \sum_{j \in \mathbf{Z}} \left| \mathcal{F}(\gamma_{j+\frac{1}{2}}, U_j^n) - \mathcal{F}(\gamma_{j-\frac{1}{2}}, U_j^n) \right| \\ &\leq \sum_{j \in \mathbf{Z}} \left| \Delta_-^u \mathcal{F}(\gamma_{j-\frac{1}{2}}, U_j^n) \right| + \|\mathcal{F}_u\| TV(\gamma), \end{aligned}$$

yielding the desired variation bound as a result of Lemma 3.7.

Using the fact that  $u^\Delta \in [0, 1]$  and the definition (3.1) of  $\mathcal{F}$  provides an  $L^\infty$  bound:  $\|\mathcal{F}^\Delta\| \leq \|f_u\|$ . To show that  $\mathcal{F}^\Delta(\cdot, t)$  is in  $L^1(\mathbf{R})$  (uniformly) on each of the time slices  $0 \leq t \leq T$ , observe that  $\mathcal{F}(\gamma, 0) = 0$ , and so, with  $n$  chosen so that  $t \in [t^n, t^{n+1})$ ,

$$\begin{aligned}
 \int_{\mathbf{R}} |\mathcal{F}^\Delta(x, t)| dx &= \int_{\mathbf{R}} |\mathcal{F}^\Delta(x, t^n)| dx \\
 &= \int_{\mathbf{R}} |\mathcal{F}(\gamma^\Delta(x), u^\Delta(x, t^n))| dx \\
 &= \int_{\mathbf{R}} |\mathcal{F}(\gamma^\Delta(x), u^\Delta(x, t^n)) - \mathcal{F}(\gamma^\Delta(x), 0)| dx \\
 &\leq \|f_u\| \int_{\mathbf{R}} |u^\Delta(x, t^n) - 0| dx \\
 (3.27) \quad &\leq \|f_u\| \left( C_3 T + \|u_0\|_{L^1(\mathbf{R})} + \frac{1}{2} \Delta x TV(u_0) \right),
 \end{aligned}$$

by Lemma 3.4. For time continuity, Lipschitz continuity of  $\mathcal{F}$  with respect to its second argument gives

$$\begin{aligned}
 &\int_{\mathbf{R}} |\mathcal{F}^\Delta(x, t^n + \Delta t) - \mathcal{F}^\Delta(x, t^n)| dx \\
 &= \frac{\Delta x}{2} \sum_{j \in \mathbf{Z}} \left( |\mathcal{F}(\gamma_{j+\frac{1}{2}}, U_j^{n+1}) - \mathcal{F}(\gamma_{j+\frac{1}{2}}, U_j^n)| + |\mathcal{F}(\gamma_{j-\frac{1}{2}}, U_j^{n+1}) - \mathcal{F}(\gamma_{j-\frac{1}{2}}, U_j^n)| \right) \\
 &\leq \|\mathcal{F}_u\| \Delta x \sum_{j \in \mathbf{Z}} |U_j^{n+1} - U_j^n| \leq C_3 \|\mathcal{F}_u\| \Delta t,
 \end{aligned}$$

by Lemma 3.3. Using this estimate it is easy to check that

$$(3.28) \quad \|\mathcal{F}^\Delta(\cdot, t + \tau) - \mathcal{F}^\Delta(\cdot, t)\|_{L^1(\mathbf{R})} \leq C_3 \|\mathcal{F}_u\| (|\tau| + \Delta t).$$

Let  $\{X_k\}_{k=1}^\infty$  be a sequence of positive numbers with  $X_k \rightarrow \infty$ . From standard compactness results (Lemma 16.8 of [42]), there is a subsequence (which we do not bother to relabel) of  $\{\mathcal{F}^\Delta\}$  such that for any fixed  $X_k$ ,

$$(3.29) \quad \int_{-X_k}^{X_k} |\mathcal{F}^\Delta(x, t) - \overline{\mathcal{F}}(x, t)| dx \rightarrow 0, \quad t \in [0, T], \quad \text{and}$$

$$(3.30) \quad \int_0^T \int_{-X_k}^{X_k} |\mathcal{F}^\Delta(x, t) - \overline{\mathcal{F}}(x, t)| dx dt \rightarrow 0,$$

for some measurable function  $\overline{\mathcal{F}}$ . By passing to a further subsequence if necessary, we may assume that  $\mathcal{F}^\Delta$  converges a.e. in  $[-X_k, X_k] \times [0, T]$ . By applying a standard diagonal process, we can find a subsequence (which we do not bother to relabel) such that  $\mathcal{F}^\Delta$  converges to  $\overline{\mathcal{F}}$  a.e. in  $\mathbf{R} \times [0, T]$ . Note that (3.29) and (3.30) remain valid for our subsequence. To show that each  $\overline{\mathcal{F}}(\cdot, t) \in L^1(\mathbf{R})$ , an application of the triangle inequality yields, for any  $X_k$ ,

$$\begin{aligned}
 \int_{-X_k}^{X_k} |\overline{\mathcal{F}}(x, t)| dx &\leq \int_{-X_k}^{X_k} |\overline{\mathcal{F}}(x, t) - \mathcal{F}^\Delta(x, t)| dx + \int_{-X_k}^{X_k} |\mathcal{F}^\Delta(x, t)| dx \\
 &\leq \int_{-X_k}^{X_k} |\overline{\mathcal{F}}(x, t) - \mathcal{F}^\Delta(x, t)| dx + \|f_u\| \left( C_3 T + \|u_0\|_{L^1(\mathbf{R})} + \frac{1}{2} \Delta x TV(u_0) \right).
 \end{aligned}$$

Letting first  $\Delta \rightarrow 0$  and then  $X_k \rightarrow \infty$ , and applying (3.29), we have

$$\|\overline{\mathcal{F}}(\cdot, t)\|_{L^1(\mathbf{R})} \leq \|f_u\| \left( C_3 T + \|u_0\|_{L^1(\mathbf{R})} \right),$$

proving that  $\overline{\mathcal{F}}(\cdot, t) \in L^1(\mathbf{R})$ . It is clear from this estimate that

$$\|\overline{\mathcal{F}}\|_{L^1(\Pi_T)} \leq T \|f_u\| \left( C_3 T + \|u_0\|_{L^1(\mathbf{R})} \right),$$

and so  $\overline{\mathcal{F}} \in L^1(\Pi_T)$ . That  $\overline{\mathcal{F}} \in L^\infty(\Pi_T)$  is clear from the bound on  $\mathcal{F}^\Delta$ .  $\square$



To show strong compactness of  $A(u^\Delta)$ , we shall in the following three lemmas obtain uniform  $L^2(\Pi_T)$  estimates on the space and time translates of  $A(u^\Delta)$ .

**Lemma 3.9.** *Assume that the hypotheses of Lemma 3.2 are satisfied. There exists a constant  $C_{11}$ , independent of  $\Delta$ , such that*

$$(3.31) \quad \left\{ \Delta t \Delta x \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} (\Delta_+ A(U_j^n))^2 \right\}^{\frac{1}{2}} \leq C_{11} \Delta x.$$

*Proof.* The proof is a discrete energy argument similar to the one used in [29] (but the proof here is a bit simpler since we know that the difference approximations are  $L^1$  Lipschitz continuous in time). Multiplying (2.4) by  $\Delta t \Delta x U_j^n$ , summing over  $n, j$ , and doing summation by parts in  $j$ , we find that

$$\begin{aligned} \Delta x \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} U_j^n (U_j^{n+1} - U_j^n) + \Delta t \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} U_j^n \Delta_- h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) \\ + \frac{\Delta t}{\Delta x} \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} \Delta_+ U_j^n \Delta_+ A(U_j^n) = 0. \end{aligned}$$

Observe that we can write

$$U_j^n (U_j^{n+1} - U_j^n) = \frac{1}{2} \left( (U_j^{n+1})^2 - (U_j^n)^2 - (U_j^{n+1} - U_j^n)^2 \right).$$

Since  $A'(\cdot) \geq 0$ , we also have

$$\frac{1}{\|A'\|} (\Delta_+ A(U_j^n))^2 \leq (\Delta_+ U_j^n) (\Delta_+ A(U_j^n)).$$

Using these observations in (3) as well as (2.2), (3.12) and Lemma 3.3, we get

$$\begin{aligned} \Delta t \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} U_j^n \Delta_+^u h_-(\gamma_{j+\frac{1}{2}}, U_j^n) + \Delta t \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} U_j^n \Delta_-^u h_+(\gamma_{j+\frac{1}{2}}, U_j^n) \\ + \frac{1}{\|A'\|} \frac{\Delta t}{\Delta x} \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} (\Delta_+ A(U_j^n))^2 \\ \leq \frac{-\Delta x}{2} \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} \left( (U_j^{n+1})^2 - (U_j^n)^2 - (U_j^{n+1} - U_j^n)^2 \right) + C_{12} T \cdot TV(\gamma) \\ \leq \frac{\Delta x}{2} \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} (U_j^{n+1} - U_j^n)^2 + \frac{\Delta x}{2} \sum_{j \in \mathbf{Z}} (U_j^0)^2 + C_{12} T \cdot TV(\gamma) \\ \leq \max_{n,j} |U_j^n| \Delta x \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} |U_j^{n+1} - U_j^n| + \frac{\Delta x}{2} \sum_{j \in \mathbf{Z}} (U_j^0)^2 + C_{12} T \cdot TV(\gamma) \leq C_{13}, \end{aligned} \quad (3.32)$$

for some finite constant  $C_{13}$  that is independent of  $\Delta$ .

To continue our analysis, we need to introduce the functions

$$\mathcal{H}_\pm(\gamma, \xi) = \int_0^\xi w \partial_w h_\pm(\gamma, w) dw = \xi h_\pm(\gamma, \xi) - \int_0^\xi h_\pm(\gamma, \xi) d\xi.$$

Then the following equalities hold

$$\begin{aligned} U_j^n \Delta_+^u h_-(\gamma_{j+\frac{1}{2}}, U_j^n) &= \Delta_+^u \mathcal{H}_-(\gamma_{j+\frac{1}{2}}, U_j^n) - \int_{U_{j+1}^n}^{U_j^n} \left( h_-(\gamma_{j+\frac{1}{2}}, w) - h_-(\gamma_{j+\frac{1}{2}}, U_{j+1}^n) \right) dw, \\ U_j^n \Delta_-^u h_+(\gamma_{j+\frac{1}{2}}, U_j^n) &= \Delta_-^u \mathcal{H}_+(\gamma_{j+\frac{1}{2}}, U_j^n) + \int_{U_{j-1}^n}^{U_j^n} \left( h_+(\gamma_{j+\frac{1}{2}}, w) - h_+(\gamma_{j+\frac{1}{2}}, U_{j-1}^n) \right) dw. \end{aligned}$$

Hence

$$\begin{aligned}
(3.33) \quad \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} U_j^n \Delta_+^u h_-(\gamma_{j+\frac{1}{2}}, U_j^n) &= - \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} \int_{U_{j+1}^n}^{U_j^n} \left( h_-(\gamma_{j+\frac{1}{2}}, w) - h_-(\gamma_{j+\frac{1}{2}}, U_{j+1}^n) \right) dw \\
&\quad + \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} \left( \Delta_+^u - \Delta_+ \right) \mathcal{H}_-(\gamma_{j+\frac{1}{2}}, U_j^n), \\
\sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} U_j^n \Delta_-^u h_+(\gamma_{j+\frac{1}{2}}, U_j^n) &= \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} \int_{U_{j-1}^n}^{U_j^n} \left( h_+(\gamma_{j+\frac{1}{2}}, w) - h_+(\gamma_{j+\frac{1}{2}}, U_{j-1}^n) \right) dw \\
&\quad + \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} \left( \Delta_-^u - \Delta_- \right) \mathcal{H}_+(\gamma_{j+\frac{1}{2}}, U_j^n),
\end{aligned}$$

and

$$\begin{aligned}
(3.34) \quad \Delta t \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} \left| \left( \Delta_-^u - \Delta_- \right) \mathcal{H}_+(\gamma_{j+\frac{1}{2}}, U_j^n) \right| &\leq C_{14} T \cdot TV(\gamma), \\
\Delta t \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} \left| \left( \Delta_+^u - \Delta_+ \right) \mathcal{H}_-(\gamma_{j-\frac{1}{2}}, U_j^n) \right| &\leq C_{14} T \cdot TV(\gamma).
\end{aligned}$$

To bound the terms involving integrals, we need the following technical result (an easy proof can be found in [19]): Let  $g : \mathbf{R} \rightarrow \mathbf{R}$  be a monotone Lipschitz continuous function with Lipschitz constant  $L_g$ . Then we have

$$\left| \int_{\xi_1}^{\xi_2} (g(w) - g(\xi_1)) dw \right| \geq \frac{1}{2L_g} (g(\xi_2) - g(\xi_1))^2, \quad \forall \xi_1, \xi_2 \in \mathbf{R}.$$

Applying this to  $h_-, h_+$  we find that

$$\begin{aligned}
(3.35) \quad - \int_{U_{j+1}^n}^{U_j^n} \left( h_-(\gamma_{j+\frac{1}{2}}, w) - h_-(\gamma_{j+\frac{1}{2}}, U_{j+1}^n) \right) dw &\geq \frac{1}{2\|f_u\|} (\Delta_+^u h_-(\gamma_{j+\frac{1}{2}}, U_j^n))^2, \\
\int_{U_{j-1}^n}^{U_j^n} \left( h_+(\gamma_{j+\frac{1}{2}}, w) - h_+(\gamma_{j+\frac{1}{2}}, U_{j-1}^n) \right) dw &\geq \frac{1}{2\|f_u\|} (\Delta_-^u h_+(\gamma_{j+\frac{1}{2}}, U_j^n))^2.
\end{aligned}$$

Inserting (3.35) into (3.32) yields via (3.33) the following inequality:

$$\begin{aligned}
(3.36) \quad \frac{\Delta t}{2\|f_u\|} \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} (\Delta_+^u h_-(\gamma_{j+\frac{1}{2}}, U_j^n))^2 &+ \frac{\Delta t}{2\|f_u\|} \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} (\Delta_-^u h_+(\gamma_{j+\frac{1}{2}}, U_j^n))^2 \\
&+ \frac{1}{\|A'\|} \frac{\Delta t}{\Delta x} \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} (\Delta_+ A(U_j^n))^2 \leq C_1 + C_{12} T \cdot TV(\gamma) + 2C_{14} T \cdot TV(\gamma).
\end{aligned}$$

From (3.36), we conclude that (3.31) holds.  $\square$

**Remark 3.1.** Although we did not need it in the proof of Lemma 3.9, (3.36) also provides us with the estimates

$$(3.37) \quad \left\{ \Delta t \Delta x \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} (\Delta_+^u h_-(\gamma_{j+\frac{1}{2}}, U_j^n))^2 \right\}^{\frac{1}{2}}, \quad \left\{ \Delta t \Delta x \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} (\Delta_-^u h_+(\gamma_{j+\frac{1}{2}}, U_j^n))^2 \right\}^{\frac{1}{2}} \leq C_{15} \sqrt{\Delta x},$$

which imply, for any  $X > 0$ , an estimate of the type

$$(3.38) \quad \Delta t \Delta x \sum_{n \in \mathbf{Z}_N^+} \sum_{|j| \leq X/\Delta x} \left( |\Delta_+^u h_-(\gamma_{j+\frac{1}{2}}, U_j^n)| + |\Delta_-^u h_+(\gamma_{j+\frac{1}{2}}, U_j^n)| \right) \leq C_{16}(X) \sqrt{\Delta x},$$

where  $C_{16}(X)$  is a constant depending on  $X$  but not  $\Delta$ . This is a sort of variation bound and for that reason the estimates in (3.37) are sometimes called weak  $BV$  estimates in the literature [7, 19, 36]. We mention that under a stronger CFL condition it is possible to prove Lemma 3.10 without using  $L^1$  Lipschitz continuity in time of the approximate solutions as stated in Lemma 3.3, see [29].

**Lemma 3.10.** *Assume that the hypotheses of Lemma 3.2 are satisfied. There exists a constant  $C_{17}$ , independent of  $\Delta$ , such that*

$$\|A(u^\Delta(\cdot + y, \cdot)) - A(u^\Delta(\cdot, \cdot))\|_{L^2(\Pi_T)} \leq C_{17} \sqrt{|y|(|y| + \Delta x)}, \quad \forall y \in \mathbf{R}.$$

*Proof.* Let  $\mathcal{I}(x) \in \mathbf{Z}$  be the integer such that  $x \in I_{\mathcal{I}(x)}$ . For  $x, y \in \mathbf{R}$  and  $\ell \in \mathbf{Z}$ , let  $\chi(x, \ell) = 1$  whenever the line segment from  $x$  to  $x + y$  intersects both  $I_\ell, I_{\ell+1}$  if  $y > 0$  or both  $I_\ell, I_{\ell-1}$  if  $y < 0$ , otherwise let  $\chi(x, \ell)$  be zero. Observe that

$$\Delta x \sum_{\ell \in \mathbf{Z}} \chi(x, \ell) \leq |y| + \Delta x$$

and

$$\int_{\mathbf{R}} \chi(x, \ell) dx \leq |y|.$$

The last estimate is true since  $\int_{\mathbf{R}} \chi(x, \ell) dx$  is the measure of the set

$\{x \in \mathbf{R}: \text{the line segment from } x \text{ to } x + y \text{ intersects both } I_\ell, I_{\ell+1} \text{ if } y > 0 \text{ or both } I_\ell, I_{\ell-1} \text{ if } y < 0\}.$

Equipped with these estimates on  $\chi(x, \ell)$  and Lemma 3.9, we calculate as follows:

$$\begin{aligned} & \iint_{\Pi_T} \left( A(u^\Delta(x + y, t)) - A(u^\Delta(x, t)) \right)^2 dt dx \\ &= \Delta t \sum_{n \in \mathbf{Z}_N^+} \int_{\mathbf{R}} \left( A(u^\Delta(x + y, t_n)) - A(u^\Delta(x, t_n)) \right)^2 dx \\ &= \Delta t \sum_{n \in \mathbf{Z}_N^+} \int_{\mathbf{R}} \left( A(U_{\mathcal{I}(x+y)}^n) - A(U_{\mathcal{I}(x)}^n) \right)^2 dx \\ &= \Delta t \sum_{n \in \mathbf{Z}_N^+} \int_{\mathbf{R}} \left( \sum_{\ell \in \mathbf{Z}} \Delta_+ A(U_\ell^n) \chi(x, \ell) \right)^2 dx \\ &\leq \Delta t \sum_{n \in \mathbf{Z}_N^+} \int_{\mathbf{R}} \Delta x \sum_{\ell \in \mathbf{Z}} \chi(x, \ell) \frac{1}{\Delta x} \sum_{\ell \in \mathbf{Z}} \left( \Delta_+ A(U_\ell^n) \right)^2 \chi(x, \ell) dx \\ &\leq (|y| + \Delta x) \frac{1}{\Delta x} \Delta t \sum_{n \in \mathbf{Z}_N^+} \sum_{\ell \in \mathbf{Z}} \left( \Delta_+ A(U_{j+\ell}^n) \right)^2 \int_{\mathbf{R}} \chi(x, \ell) dx, \\ &\leq C_{11}^2 |y| (|y| + \Delta x), \end{aligned}$$

where  $C_{11}$  is the constant in Lemma 3.9. This concludes the proof of the lemma.  $\square$

**Lemma 3.11.** *Assume that the hypotheses of Lemma 3.2 are satisfied. There exists a constant  $C_{18}$ , independent of  $\Delta$ , such that*

$$\|A(u^\Delta(\cdot, \cdot + \tau)) - A(u^\Delta(\cdot, \cdot))\|_{L^2(\Pi_{T-\tau})} \leq C_{18} \sqrt{\tau + \Delta t}, \quad \forall \tau \in (0, T).$$

*Proof.* From Lemma 3.3 it follows that

$$\int_{\mathbf{R}} |u^\Delta(x, t + \tau) - u^\Delta(x, t)| dx \leq C_3(\tau + \Delta t),$$

for all  $t, \tau$  such that  $t, t + \tau \in (0, T)$ . Now “interpolating between  $L^1$  and  $L^\infty$ ”, we obtain the desired result:

$$\begin{aligned} & \int_0^{T-\tau} \int_{\mathbf{R}} (A(u^\Delta(x, t + \tau)) - A(u^\Delta(x, t)))^2 dx dt \\ & \leq C_{19} \int_0^{T-\tau} \int_{\mathbf{R}} |u^\Delta(x, t + \tau) - u^\Delta(x, t)| dx dt \leq C_{20}(\tau + \Delta t), \end{aligned}$$

where the constants  $C_{19}$  and  $C_{20}$  do not depend on  $\Delta$ .  $\square$

**Remark 3.2.** Again under a stronger CFL condition, it is possible to prove Lemma 3.11 without using  $L^1$  Lipschitz continuity in time of the approximate solutions (Lemma 3.3), see [29].

**Lemma 3.12.** *Assume that the hypotheses of Lemma 3.2 are satisfied. There exists a subsequence of  $\{A^\Delta\}$ , also denoted by  $\{A^\Delta\}$ , and a function*

$$(3.39) \quad \bar{A} \in L^2(0, T; H^1(\mathbf{R}))$$

*such that  $A^\Delta \rightarrow \bar{A}$  in  $L^2_{\text{loc}}(\Pi_T)$  and a.e. in  $\Pi_T$ . Furthermore,  $\bar{A} = A(u)$  a.e. in  $\Pi_T$ , where  $u$  denotes the  $L^\infty$  weak-\* limit of  $u^\Delta$ .*

*Proof.* For the compactness part of the lemma, one has to use Lemmas 3.10 and 3.11 and repeat the proof of Kolmogorov’s  $L^p$  compactness criterion (see Theorem IV.8.21 in [12]). We omit the tedious but straightforward details. In view of Lemma 3.10 and the strong convergence  $A^\Delta \rightarrow \bar{A}$ , we clearly have

$$\iint_{\Pi_T} (\bar{A}(x + y, t) - \bar{A}(x, t))^2 dt dx \leq C_{17}^2 |y|^2,$$

for any  $y \in \mathbf{R}$ . From this it follows that (3.39) holds, and the  $L^2(0, T; H^1(\mathbf{R}))$  norm of  $\bar{A}$  is bounded by  $C_{17}$ . The proof of the final part of the lemma can be found in [31].  $\square$

We are now in a position to prove our main theorem.

**Theorem 3.1.** *Assume that the hypotheses concerning the data stated in Section 1 are satisfied. Let  $u^\Delta$  be defined by (2.8) and the scheme (2.4), with the parameters  $\lambda$  and  $\mu$  chosen so that the CFL condition (3.3) holds. Then there exists a weak solution  $u$  of the initial value problem (1.1) in the sense of Definition 1.1, and a subsequence of  $\{u^\Delta\}$ , also denoted by  $\{u^\Delta\}$ , such that  $u^\Delta \rightarrow u$  in  $L^1_{\text{loc}}(\Pi_T)$  and a.e. in  $\Pi_T$ .*

*Proof.* Let  $z^\Delta = \Psi(\gamma^\Delta, u^\Delta) = \mathcal{F}^\Delta + A^\Delta$ . Both of the sequences  $\{\mathcal{F}^\Delta\}$  and  $\{A^\Delta\}$  have subsequences converging boundedly a.e. in  $\Pi_T$ , by Lemma 3.8 and Lemma 3.12, and therefore in  $L^1_{\text{loc}}(\Pi_T)$ . By passing to a further subsequence on which both  $\{\mathcal{F}^\Delta\}$  and  $\{A^\Delta\}$  converge, there is a subsequence, also denoted by  $\{z^\Delta\}$ , such that for some  $z \in L^1_{\text{loc}}(\Pi_T) \cap L^\infty(\Pi_T)$ ,  $z^\Delta \rightarrow z$  in  $L^1_{\text{loc}}(\Pi_T)$  and a.e. Let  $u(x, t) = \Psi^{-1}(\gamma(x), z(x, t))$ , which is well-defined a.e. in  $\Pi_T$ , thanks to the fact that  $\Psi(\gamma, w)$  is strictly increasing as a function of  $w$ . The immediate goal is to show that we have pointwise convergence of  $u^\Delta$  a.e. in  $\Pi_T$ . Suppressing the dependence on the point  $(x, t)$ ,

$$\begin{aligned} |\Psi(\gamma, u^\Delta) - \Psi(\gamma, u)| & \leq |\Psi(\gamma, u^\Delta) - \Psi(\gamma^\Delta, u^\Delta)| + |\Psi(\gamma^\Delta, u^\Delta) - \Psi(\gamma, u)| \\ & \leq \|\Psi_\gamma\| |\gamma - \gamma^\Delta| + |z^\Delta - z|. \end{aligned}$$

Thus, since  $\gamma^\Delta \rightarrow \gamma$  a.e. and  $z^\Delta \rightarrow z$  a.e.,  $\Psi(\gamma, u^\Delta) \rightarrow \Psi(\gamma, u)$  a.e. in  $\Pi_T$ . Since  $\Psi(\gamma, \cdot)$  is strictly increasing, it follows that  $u^\Delta \rightarrow u$  boundedly a.e., from which convergence in  $L^1_{\text{loc}}(\Pi_T)$  follows.

Since each  $u^\Delta \in [0, 1]$ , it is clear that  $u \in L^\infty(\Pi_T)$ . Also, it is immediate from Lemma 3.12 that  $A(u) \in L^2(0, T; H^1(\mathbf{R}))$ . To prove that  $u \in L^1(\Pi_T)$ , fix  $X > 0$ , and set  $\Pi_T^X = [-X, X] \times [0, T]$ . Then

$$\iint_{\Pi_T^X} |u(x, t)| dt dx \leq \iint_{\Pi_T^X} |u(x, t) - u^\Delta(x, t)| dt dx + \iint_{\Pi_T^X} |u^\Delta(x, t)| dt dx$$

$$\leq \iint_{\Pi_T^X} |u(x, t) - u^\Delta(x, t)| \, dt \, dx + \iint_{\Pi_T} |u^\Delta(x, t)| \, dt \, dx.$$

After invoking Lemma 3.4, then letting  $\Delta \rightarrow 0$  and subsequently  $X \rightarrow \infty$ , we get

$$\iint_{\Pi_T} |u(x, t)| \, dt \, dx \leq T \left( C_3 T + \|u(\cdot, 0)\|_{L^1(\mathbf{R})} \right),$$

proving that  $u \in L^1(\Pi_T)$ . As a result of the time continuity estimate of Lemma 3.3, and by passing to a further subsequence if necessary,  $u^\Delta(\cdot, t) \rightarrow u(\cdot, t)$  in  $L^1(\mathbf{R})$  for each  $t \in [0, T]$  (see, e.g., the proof of Lemma 16.8 of [42]). To show that  $u \in C(0, T; L^1(\mathbf{R}))$ , let  $\tau > 0$ , and apply the triangle inequality:

$$\begin{aligned} \|u(\cdot, t + \tau) - u(\cdot, t)\|_{L^1(\mathbf{R})} &\leq \|u(\cdot, t + \tau) - u^\Delta(\cdot, t + \tau)\|_{L^1(\mathbf{R})} \\ &\quad + \|u^\Delta(\cdot, t + \tau) - u^\Delta(\cdot, t)\|_{L^1(\mathbf{R})} + \|u^\Delta(\cdot, t) - u(\cdot, t)\|_{L^1(\mathbf{R})}. \end{aligned}$$

It is a simple consequence of Lemma 3.3 that

$$\|u^\Delta(\cdot, t + \tau) - u^\Delta(\cdot, t)\|_{L^1(\mathbf{R})} \leq C_3(\tau + \Delta t).$$

Using this fact, and letting  $\Delta \rightarrow 0$  in the preceding inequality gives the desired  $L^1$  time continuity estimate for the limit function  $u$ , proving that  $u \in C(0, T; L^1(\mathbf{R}))$ .

It remains to show that the limit solution  $u$  is a weak solution to the initial value problem (1.1), for which a version of the Lax-Wendroff theorem is required. Let  $\phi \in \mathcal{D}(\Pi_T)$  with  $\phi|_{t=T} = 0$ . Fix  $X > 0$  such that  $\phi$  vanishes for  $|x| \geq X$ . Let  $\phi_j^n = \phi(x_j, t^n)$ . Multiplying the difference scheme (2.4) by  $\phi_j^n \Delta x$ , then summing by parts results in

$$\begin{aligned} & -\Delta x \Delta t \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} U_j^n \frac{\phi_j^n - \phi_j^{n-1}}{\Delta t} - \Delta x \sum_{j \in \mathbf{Z}} U_j^0 \phi_j^0 \\ & - \Delta x \Delta t \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) \frac{1}{\Delta x} \Delta_+ \phi_j^n + \Delta x \Delta t \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} A(U_j^n) \frac{1}{\Delta x^2} \Delta_- \Delta_+ \phi_j^n = 0, \end{aligned}$$

where  $J\Delta x = X$  and  $N = \lfloor T/Dt \rfloor$ , and  $j \in \{-J, \dots, J\}$ ,  $n \in \{0, \dots, N\}$ . For  $(x, t) \in R_j^n$  we have that

$$\left. \begin{aligned} \frac{\phi_j^n - \phi_j^{n-1}}{\Delta t} &= \phi_t(x, t) \\ \frac{1}{\Delta x} \Delta_+ \phi_j^n &= \phi_x(x, t) \\ \frac{1}{\Delta x^2} \Delta_- \Delta_+ \phi_j^n &= \phi_{xx}(x, t) \end{aligned} \right\} + \mathcal{O}(\Delta t + \Delta x),$$

and

$$\phi_j^0 = \phi(x, 0) + \mathcal{O}(\Delta x).$$

Therefore we have that

$$\begin{aligned} (3.40) \quad & \Delta x \Delta t \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} U_j^n \frac{\phi_j^n - \phi_j^{n-1}}{\Delta t} + \Delta x \Delta t \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} A(U_j^n) \frac{1}{\Delta x^2} \Delta_- \Delta_+ \phi_j^n + \Delta x \sum_{j \in \mathbf{Z}} U_j^0 \phi_j^0 \\ & = \iint_{\Pi_T} \left( u^\Delta \phi_t + A(u^\Delta) \phi_{xx} \right) dt \, dx + \int_{\mathbf{R}} u^\Delta(x, 0) \phi(x, 0) \, dx + \mathcal{O}(\Delta x + \Delta t). \end{aligned}$$

Now defining  $\gamma_j = \frac{1}{2}(\gamma(x_j-) + \gamma(x_j+))$  and defining  $\tilde{\gamma}^\Delta$  as

$$\tilde{\gamma}^\Delta(x) = \sum_{j \in \mathbf{Z}} \chi_j(x) \gamma_j,$$

we compute

$$h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) - f(\gamma_j, U_j^n) = h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) - h(\gamma_j, U_{j+1}^n, U_j^n)$$

$$\begin{aligned}
& + h(\gamma_j, U_{j+1}^n, U_j^n) - h(\gamma_j, U_j^n, U_j^n) \\
& = \int_{\gamma_j}^{\gamma_{j+\frac{1}{2}}} h_\gamma(w, U_{j+1}^n, U_j^n) dw + \int_{U_j^n}^{U_{j+1}^n} f_u^-(\gamma_j, w) dw \\
& = \int_{\gamma_j}^{\gamma_{j+\frac{1}{2}}} h_\gamma(w, U_{j+1}^n, U_j^n) dw + \Delta_+^u h_-(\gamma_j, U_j^n).
\end{aligned}$$

Consequently,

$$\begin{aligned}
& \Delta x \Delta t \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) \frac{1}{\Delta x} \Delta_- \phi_j^n \\
& = \iint_{\Pi_T} f(\tilde{\gamma}^\Delta(x), u^\Delta) \phi_x(x, t) dt dx + \mathcal{O}(\Delta x) \\
& \quad + \Delta x \mathcal{O}(TV(\gamma)) + \Delta x \Delta t \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} \Delta_+^u h_-(\gamma_{j+\frac{1}{2}}, U_j^n) \frac{1}{\Delta x} \Delta_- \phi_j^n.
\end{aligned}$$

Now, using (3.38), we find that the last term above can be bounded by

$$(3.41) \quad C_\phi \sqrt{\Delta x},$$

where the constant  $C_\phi$  depends on  $\phi$  but not on  $\Delta$ . Collecting these bounds we find that

$$(3.42) \quad \iint_{\Pi_T} \left( u^\Delta \phi_t + f(\tilde{\gamma}^\Delta(x), u^\Delta) \phi_x + A(u^\Delta) \phi_{xx} \right) dt dx + \int_{\mathbf{R}} u^\Delta(x, 0) \phi(x, 0) dx = \mathcal{O}(\sqrt{\Delta x}),$$

where the  $\mathcal{O}()$  term on the right depends only on  $\phi$ . Since  $A(u) \in L^2(0, T; H^1(\mathbf{R}))$  it is possible to integrate by parts in  $x$ , and hence

$$\iint_{\Pi_T} A(u^\Delta) \phi_{xx} dt dx \rightarrow \iint_{\Pi_T} A(u) \phi_{xx} dt dx = - \iint_{\Pi_T} A(u)_x \phi_x dt dx.$$

Letting  $\Delta \downarrow 0$  in (3.42), we thus find that  $u$  is a weak solution, and that we have a “weak convergence rate” of  $1/2$ .  $\square$

**Remark 3.3.** Note that  $TV_{[-X, X]} z^\Delta(\cdot, t) \leq C_{21}$ , for some constant  $C_{21}$  that is independent of  $\Delta$  but dependent on  $X$ . Here the  $R$  dependence comes from only having an  $L^2$  space translation estimate on  $A(u^\Delta)$ . This bound could have been used directly to get strong compactness of  $z^\Delta$ .

#### 4. ADDITIONAL REGULARITY

In this section we show that  $A(u)$ , where  $u$  is the limit constructed in Theorem 3.1, can be identified a.e. with a  $C^{1, \frac{1}{2}}(\Pi_T)$  function. To this end, we will in the next two lemmas obtain uniform  $L^\infty$  estimates of the space and time translates of  $\{A(U_j^n)\}$ .

**Lemma 4.1.** *With the hypotheses of Theorem 3.1, there exists a constant  $C_{22}$ , independent of  $\Delta$ , such that*

$$|A(U_j^n) - A(U_i^n)| \leq C_{22} |j - i| \Delta x, \quad \forall i, j \in \mathbf{Z}.$$

*Proof.* Assuming that  $M$  is a positive integer such that  $U_j^n = 0$  for  $j \leq -M + 1$ , from the definition of the scheme (2.4),

$$\begin{aligned}
\frac{1}{\Delta x} |\Delta_+ A(U_j^n)| - |h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n)| & \leq \left| h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) - \frac{1}{\Delta x} \Delta_+ A(U_j^n) \right| \\
& = \left| \sum_{\ell=-M}^j \Delta_- \left( h(\gamma_{\ell+\frac{1}{2}}, U_{\ell+1}^n, U_\ell^n) - \frac{1}{\Delta x} \Delta_+ A(U_\ell^n) \right) \right| \\
& = \frac{1}{\lambda} \left| \sum_{\ell=-M}^j (U_\ell^{n+1} - U_\ell^n) \right|
\end{aligned}$$

$$\leq \frac{\Delta x}{\Delta t} \sum_{j \in \mathbf{Z}} |U_j^{n+1} - U_j^n| \leq C_3.$$

Since  $h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n)$  is bounded,

$$|\Delta_+ A(U_j^n)| \leq C_{22} \Delta x,$$

and the lemma follows.  $\square$

Although the above  $L^\infty(\Pi_T)$  space translation estimate was an easy consequence of Lemma 3.3, the  $L^\infty(\Pi_T)$  time translation estimate is a bit trickier to obtain, as shown by the proof of the next lemma.

**Lemma 4.2.** *With the hypotheses of Theorem 3.1, there exists a constant  $C_{23}$ , independent of  $\Delta$ , such that*

$$|A(U_j^n) - A(U_j^m)| \leq C_{23} \sqrt{|n - m| \Delta t}.$$

*Proof.* The proof is an adaptation of a technique used in [17]. To prove the lemma, we shall need to work with an interpolant of the discrete values  $\{U_j^n\}$  that is continuous everywhere and differentiable almost everywhere. For this purpose, define  $\tilde{u}^n(x)$  as

$$\tilde{u}^n(x) = \frac{1}{\Delta x} \left( (x - x_{j-1}) U_j^n + (x_j - x) U_{j-1}^n \right), \quad x \in [x_{j-1}, x_j].$$

Then define

$$\tilde{u}^\Delta(x, t) = \frac{1}{\Delta t} \left( (t - t_n) \tilde{u}^{n+1}(x) + (t_{n+1} - t) \tilde{u}^n(x) \right), \quad t \in [t_n, t_{n+1}].$$

As before, let  $\mathcal{I}(x) \in \mathbf{Z}$  be the integer satisfying  $x \in [x_{\mathcal{I}(x)}, x_{\mathcal{I}(x)+1})$ , so that  $\mathcal{I}(x_j + \alpha) - j =: J$  for some  $J \in \mathbf{Z}$ . Since  $\tilde{u}^\Delta$  is differentiable in time almost everywhere on  $\Pi_T$ , we can proceed as follows:

$$\begin{aligned} & \int_{x_j}^{x_j+\alpha} (\tilde{u}^\Delta(x, t_n) - \tilde{u}^\Delta(x, t_m)) \, dx \\ &= \int_{x_j}^{x_j+\alpha} \int_{t_m}^{t_n} \partial_t \tilde{u}^\Delta(x, \tau) \, d\tau \, dx \\ &= \sum_{k=j}^{\mathcal{I}(x_j+\alpha)-1} \sum_{\ell=m}^{n-1} \iint_{R_{k-1/2}^\ell} \partial_t \tilde{u}^\Delta(x, \tau) \, d\tau \, dx \\ &= \sum_{k=j}^{\mathcal{I}(x_j+\alpha)-1} \sum_{\ell=m}^{n-1} \int_{x_k}^{x_{k+1}} (\tilde{u}^{\ell+1}(x) - \tilde{u}^\ell(x)) \, dx \\ &= \sum_{k=j}^{\mathcal{I}(x_j+\alpha)-1} \sum_{\ell=m}^{n-1} \int_{x_k}^{x_{k+1}} \frac{1}{\Delta x} \left( (x - x_k) (U_{k+1}^{\ell+1} - U_{k+1}^\ell) + (x_{k+1} - x) (U_k^{\ell+1} - U_k^\ell) \right) \, dx \\ &= \frac{\Delta x}{2} \sum_{\ell=m}^{n-1} \sum_{k=j}^{\mathcal{I}(x_j+\alpha)-1} \left( (U_{k+1}^{\ell+1} - U_{k+1}^\ell) + (U_k^{\ell+1} - U_k^\ell) \right). \end{aligned}$$

Hence, using Lemma 3.3, we find that

$$(4.1) \quad \left| \int_{x_j}^{x_j+\alpha} (\tilde{u}^\Delta(x, t_n) - \tilde{u}^\Delta(x, t_m)) \, dx \right| \leq C_3 |n - m| \Delta t.$$

Now set  $\alpha = \sqrt{|n-m|\Delta t}$ . By the mean value theorem, there exists a number  $x^*$  in  $[x_j, x_j + \alpha]$  such that

$$(4.2) \quad |\tilde{u}^\Delta(x^*, t_n) - \tilde{u}^\Delta(x^*, t_m)| = \frac{1}{\alpha} \left| \int_{x_j}^{x_j + \alpha} (\tilde{u}^\Delta(\xi, t_n) - \tilde{u}^\Delta(\xi, t_m)) d\xi \right| = \mathcal{O} \left( \sqrt{|n-m|\Delta t} \right),$$

where we have used (4.1). From this we derive the following estimate:

$$(4.3) \quad \begin{aligned} E_2 &= |A(\tilde{u}^\Delta(x^*, t_n)) - A(\tilde{u}^\Delta(x^*, t_m))| \leq \|A'\| |\tilde{u}^\Delta(x^*, t_n) - \tilde{u}^\Delta(x^*, t_m)| \\ &= \mathcal{O} \left( \sqrt{|n-m|\Delta t} \right). \end{aligned}$$

By the triangle inequality,

$$|A(U_j^n) - A(U_j^m)| = |A(\tilde{u}^\Delta(x_j, t_n)) - A(\tilde{u}^\Delta(x_j, t_m))| \leq E_1 + E_2 + E_3,$$

where

$$\begin{aligned} E_1 &= |A(\tilde{u}^\Delta(x_j, t_n)) - A(\tilde{u}^\Delta(x^*, t_n))|, \\ E_2 &= |A(\tilde{u}^\Delta(x^*, t_n)) - A(\tilde{u}^\Delta(x^*, t_m))|, \\ E_3 &= |A(\tilde{u}^\Delta(x^*, t_m)) - A(\tilde{u}^\Delta(x_j, t_m))|. \end{aligned}$$

By Lemma 4.1,

$$E_1 + E_3 = \mathcal{O}(|x^* - x_j|) = \mathcal{O}(\alpha) = \mathcal{O} \left( \sqrt{|n-m|\Delta t} \right),$$

which finishes the proof of the lemma.  $\square$

Introduce the piecewise bilinear interpolant  $A^\Delta(x, t)$  interpolating the values  $A(U_j^n)$  in the same way that  $\tilde{u}^\Delta$  interpolates the values  $U_j^n$ . Then the main theorem of this section can be stated as follows:

**Theorem 4.1.** *With the hypotheses of Theorem 3.1, there exists a subsequence of  $A^\Delta$ , also denoted by  $A^\Delta$ , and a function*

$$\bar{A} \in C^{1, \frac{1}{2}}(\Pi_T)$$

*such that  $A^\Delta \rightarrow \bar{A}$  in  $L_{\text{loc}}^\infty(\Pi_T)$ . Furthermore,  $\bar{A} = A(u)$  a.e. in  $\Pi_T$ , where  $u$  is the weak solution constructed in Theorem 3.1.*

*Proof.* Let  $(j, n)$  and  $(i, m)$  be integers such that  $(x, t) \in R_{j+\frac{1}{2}}^n$  and  $(x+y, t+\tau) \in R_{i+\frac{1}{2}}^m$ . Then

$$|A^\Delta(x+y, t+\tau) - A^\Delta(x, t)| \leq E_1 + E_2 + E_3,$$

where

$$\begin{aligned} E_1 &= |A^\Delta(x+y, t+\tau) - A^\Delta(x_i, t_m)|, \\ E_2 &= |A^\Delta(x_i, t_m) - A^\Delta(x_j, t_n)|, \\ E_3 &= |A^\Delta(x_j, t_n) - A^\Delta(x, t)|. \end{aligned}$$

Lemmas 4.1 and 4.2 imply that  $E_2 = \mathcal{O}(|i-j|\Delta x + \sqrt{|n-m|\Delta t})$ . Obviously, for  $(x, t) \in R_{j+\frac{1}{2}}^n$  we have

$$(4.4) \quad \begin{aligned} \min \{A(U_j^n), A(U_{j+1}^n), A(U_j^{n+1}), A(U_{j+1}^{n+1})\} \\ \leq A^\Delta(x, t) \leq \max \{A(U_j^n), A(U_{j+1}^n), A(U_j^{n+1}), A(U_{j+1}^{n+1})\}. \end{aligned}$$

From this and again Lemmas 4.1 and 4.2, we have  $E_1 + E_3 = \mathcal{O}(\Delta x + \sqrt{\Delta t})$ . Thus there exists a constant  $C_{24}$ , independent of  $\Delta$ , such that

$$|A^\Delta(x+y, t+\tau) - A^\Delta(x, t)| \leq C_{24} (y + \tau + \Delta x + \sqrt{\Delta t}).$$

Equipped with this estimate, we repeat the proof of the Ascoli-Arzelà theorem to conclude that there is a subsequence of  $\{A^\Delta\}$ , still denoted by  $\{A^\Delta\}$ , and a limit function  $\bar{A} \in C^{1, \frac{1}{2}}(\Pi_T)$  such that  $A^\Delta \rightarrow \bar{A}$  uniformly on compact sets and pointwise on  $\Pi_T$ .



Next we show that  $\bar{A} = A(u)$  almost everywhere. Without loss of generality, assume for some sequence  $\Delta \rightarrow 0$  that  $u^\Delta \rightarrow u$  a.e. and  $A^\Delta \rightarrow \bar{A}$  pointwise everywhere. Pick an arbitrary but fixed point  $(x, t)$  such that  $u^\Delta(x, t) \rightarrow u(x, t)$ . We have

$$|A(u(x, t)) - \bar{A}(x, t)| \leq E_1 + E_2 + E_3,$$

where

$$\begin{aligned} E_1 &= |A(u(x, t)) - A(u^\Delta(x, t))|, \\ E_2 &= |A(u^\Delta(x, t)) - A^\Delta(x, t)|, \\ E_3 &= |A^\Delta(x, t) - \bar{A}(x, t)|. \end{aligned}$$

Obviously,  $E_1$  and  $E_3$  vanish as  $\Delta \rightarrow 0$ . Let  $j$  and  $n$  be integers such that  $(x, t) \in R_{j+\frac{1}{2}}^n$ . Then, in view of (4.4) and Lemmas 4.1 and 4.2,  $E_2 = |A(U_j^n) - A^\Delta(x, t)| = \mathcal{O}(\Delta x + \sqrt{\Delta t})$ , which also tends to zero as  $\Delta \rightarrow 0$ . This concludes the proof of the theorem.  $\square$

## 5. ENTROPY SATISFACTION

Because the diffusion term is strongly degenerate, solutions to the initial value problem (1.1) can develop discontinuities, and so solutions are not a priori unique. In the case where the coefficient  $\gamma$  is continuous, an entropy condition is used to single out the physically relevant solution. In the remainder of this section we establish a cell entropy inequality for solutions of our finite difference algorithm, and then show that limit solutions satisfy a Kruřkov-type entropy inequality. This generalizes the corresponding result in [44]. In [32], we use this entropy inequality to establish that there is an  $L^1$  contraction principle for weak solutions of the initial value problem (1.1) that are additionally assumed to be piecewise smooth.

**Lemma 5.1.** *With the hypotheses of Theorem 3.1, the following cell entropy inequality is satisfied by approximate solutions  $U_j^n$  generated by the scheme (2.4):*

(5.1)

$$V(U_j^{n+1}) \leq V(U_j^n) - \Delta_- \left( \lambda H(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) - \mu \Delta_+ |A(U_j^n) - A(c)| \right) + \lambda \left| \Delta_+ f(\gamma_{j-\frac{1}{2}}, c) \right|,$$

where the numerical entropy flux  $H(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n)$  is defined by

$$(5.2) \quad H(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) = f(\gamma_{j+\frac{1}{2}}, U_{j+1}^n \vee c, U_j^n \vee c) - f(\gamma_{j+\frac{1}{2}}, U_{j+1}^n \wedge c, U_j^n \wedge c).$$

*Proof.* The proof is an adaptation of a portion of the proof of Lemma 4.2 of [10]. Let

$$G_j(U_{j+1}^n, U_j^n, U_{j-1}^n) = U_j^{n+1} = U_j^n - \lambda \Delta_- \left( h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) - \frac{1}{\Delta x} \Delta_+ A(U_j^n) \right),$$

and observe that

$$G_j(c, c, c) = c - \lambda \Delta_- f(\gamma_{j+\frac{1}{2}}, c).$$

The following inequalities are a consequence of monotonicity of the numerical scheme:

$$(5.3) \quad G_j(c, c, c) \vee G_j(U_{j+1}^n, U_j^n, U_{j-1}^n) \leq G_j(c \vee U_{j+1}^n, c \vee U_j^n, c \vee U_{j-1}^n),$$

$$(5.4) \quad -G_j(c, c, c) \wedge G_j(U_{j+1}^n, U_j^n, U_{j-1}^n) \leq -G_j(c \wedge U_{j+1}^n, c \wedge U_j^n, c \wedge U_{j-1}^n).$$

Following [10], (5.3) and (5.4) are added, and the identity  $a \vee b - a \wedge b = |a - b|$  is applied, giving

$$(5.5) \quad \begin{aligned} &|G_j(U_{j+1}^n, U_j^n, U_{j-1}^n) - G_j(c, c, c)| \\ &\leq G_j(c \vee U_{j+1}^n, c \vee U_j^n, c \vee U_{j-1}^n) - G_j(c \wedge U_{j+1}^n, c \wedge U_j^n, c \wedge U_{j-1}^n). \end{aligned}$$

Take the left side of (5.5):

$$(5.6) \quad \begin{aligned} |G_j(U_{j+1}^n, U_j^n, U_{j-1}^n) - G_j(c, c, c)| &= \left| G_j(U_{j+1}^n, U_j^n, U_{j-1}^n) - c + \lambda \Delta_- f(\gamma_{j+\frac{1}{2}}, c) \right| \\ &\geq |U_j^{n+1} - c| - \lambda \left| \Delta_- f(\gamma_{j+\frac{1}{2}}, c) \right|. \end{aligned}$$

Now take the right side of (5.5):

$$\begin{aligned}
& G_j(c \vee U_{j+1}^n, c \vee U_j^n, c \vee U_{j-1}^n) - G_j(c \wedge U_{j+1}^n, c \wedge U_j^n, c \wedge U_{j-1}^n) \\
&= c \vee U_j^n - c \wedge U_j^n \\
&\quad - \lambda \Delta_- \left( h(\gamma_{j+\frac{1}{2}}, c \vee U_{j+1}^n, c \vee U_j^n) - h(\gamma_{j+\frac{1}{2}}, c \wedge U_{j+1}^n, c \wedge U_j^n) \right) \\
&\quad + \frac{1}{\Delta x} \lambda \Delta_- \Delta_+ (A(U_j^n \vee c) - A(U_j^n \wedge c)) \\
(5.7) \quad &= |U_j^n - c| - \lambda \Delta_- \left( H(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) - \frac{1}{\Delta x} \Delta_+ |A(U_j^n) - A(c)| \right).
\end{aligned}$$

The last step in (5.7) used the fact that  $A(U_j^n \vee c) - A(U_j^n \wedge c) = |A(U_j^n) - A(c)|$ , which results from the fact that  $A$  is nondecreasing. The proof is completed by comparing (5.6) with (5.7).  $\square$

**Theorem 5.1.** *Suppose that the hypotheses of Theorem 3.1 hold. Let  $u$  be a limit point of the sequence  $\{u^\Delta\}$  generated by the scheme (2.4) (see Theorem 3.1). The following entropy inequality holds for all nonnegative test functions  $\phi \in \mathcal{D}'(\Pi_T)$  such that  $\phi|_{t=0} = \phi|_{t=T} = 0$ :*

$$\begin{aligned}
(5.8) \quad & \iint_{\Pi_T} \left( |u - c| \phi_t + \sigma(u - c)(f(\gamma(x), u) - f(\gamma(x), c)) \phi_x + |A(u) - A(c)| \phi_{xx} \right) dt dx \\
& + \iint_{\Pi_T} |f(\gamma(x), c)_x| \phi dt dx \geq 0.
\end{aligned}$$

**Remark 5.1.** Note that in the last integral in (5.8), since  $\gamma$  is not continuous, but only of bounded variation, the term  $|f(\gamma(x), c)_x|$  must be interpreted as a measure. If we label this measure  $\nu$ , then for any set  $E \subseteq \mathbf{R}$ ,

$$\nu(E) = T.V. \big|_E (f(\gamma(\cdot), c)) = \int_E |f(\gamma(x), c)_x| dx.$$

*Proof.* Let  $V(u) = |u - c|$  and  $F(\gamma, u) = \sigma(u - c)(f(\gamma, u) - f(\gamma, c))$ . Following the proof of the Lax-Wendroff theorem, the discrete entropy inequality (5.1) is multiplied by  $\phi_j^n \Delta x$ , then summed over  $j$  and  $n$ . Here  $\phi_j^n = \phi(x_j, t^n)$  and  $\phi$  is a test function of type described in the statement of the theorem. This yields

$$\begin{aligned}
(5.9) \quad & \Delta t \Delta x \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} \phi_j^n \left( \frac{V(U_j^{n+1}) - V(U_j^n)}{\Delta t} + \frac{1}{\Delta x} \Delta_- H(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) \right. \\
(5.10) \quad & \left. - \frac{1}{\Delta x^2} \Delta_- \Delta_+ |A(U_j^n) - A(c)| \right) - \Delta t \Delta x \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} \phi_j^n \left| \frac{1}{\Delta x} \Delta_+ f(\gamma_{j-\frac{1}{2}}, c) \right| \leq 0.
\end{aligned}$$

Summing by parts and letting  $\Delta \rightarrow 0$  gives, by the bounded convergence theorem,

$$(5.11) \quad \Delta t \Delta x \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} \phi_j^n \left( \frac{V(U_j^{n+1}) - V(U_j^n)}{\Delta t} + \frac{1}{\Delta x} \Delta_- H(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) \right)$$

$$(5.12) \quad \rightarrow - \iint_{\Pi_T} \left( V(u) \phi_t + F(\gamma(x), u) \phi_x \right) dt dx,$$

as in the proof of the Lax-Wendroff theorem. For the term containing  $|A(U_j^n) - A(c)|$ , summing by parts and letting  $\Delta \rightarrow 0$  gives

$$(5.13) \quad \Delta t \Delta x \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} \phi_j^n \frac{1}{\Delta x^2} \Delta_- \Delta_+ |A(U_j^n) - A(c)| \rightarrow \iint_{\Pi_T} |A(u) - A(c)| \phi_{xx} dt dx.$$

For the remaining term,

$$(5.14) \quad \Delta t \Delta x \sum_{n \in \mathbf{Z}_N^+} \sum_{j \in \mathbf{Z}} \phi_j^n \left| \frac{1}{\Delta x} \Delta_+ f(\gamma_{j-\frac{1}{2}}, c) \right| \\ \rightarrow \iint_{\Pi_T \setminus \{\xi_m\}} |f(\gamma(x), c)_x| \phi(x, t) dt dx + \int_0^T \sum_{m=1}^{M'} |f(\gamma(\xi_m^+), c) - f(\gamma(\xi_m^-), c)| \phi(\xi_m, t) dt,$$

which follows by breaking the spatial portion of the sum into sums over intervals where  $\gamma$  is differentiable and isolating the finite number of cells where the jumps in  $\gamma$  are located. The proof is complete once (5.11), (5.13), and (5.14) are combined.  $\square$

## 6. NUMERICAL EXAMPLES

This section discusses some numerical examples for the equation

$$u_t + (\gamma(x)f(u))_x = A(u)_{xx},$$

each using the convective flux  $f(u) = u(1 - u)$ . We will focus on Riemann problems, with initial data denoted by  $(u_L, u_R)$ , meaning that  $u_0(x) = u_L$  for  $x < 0$ ,  $u_0(x) = u_R$  for  $x > 0$ . Similarly the coefficient has a single jump at the origin, which we denote by  $(\gamma_L, \gamma_R)$ . The data satisfy all of the hypotheses of the paper, with the exception that we do not have  $u_0 \in L^1(\mathbf{R})$ . This is not a serious violation, since we can imagine that  $u_0$  eventually vanishes for large  $|x|$  (thus satisfying  $u_0 \in L^1(\mathbf{R})$ ), and that the waves generated by this variation in  $u_0$  never reach the boundary of our computational domain. By choosing our output time small enough that no waves from either the interior or exterior of the computational domain ever reach the boundary, we can handle the boundary conditions by simply setting the boundary data to their initial values, and keeping them fixed throughout the computation.

**Example 1.** Figure 1 shows the result of two runs of the scheme (2.4) with the Riemann problem having constant initial data  $(u_L, u_R) = (0.6, 0.6)$ , and the coefficient given by  $(\gamma_L, \gamma_R) = (0.05, 0.1)$ . The scheme was run for 500 time steps, with  $\Delta x = .02$  and  $\Delta t = .04$ . In (a), the diffusion term was  $A(u) = 0$ , i.e., the purely hyperbolic problem. In (b), the diffusion term was

$$(6.1) \quad A(u) = .0025 \left( u \chi_{[0, .45)}(u) + .45 \chi_{[.45, .55)}(u) + (u - 0.1) \chi_{[.55, 1]}(u) \right),$$

which is degenerate ( $A'(u) = 0$ ) in the interval  $(.45, .55)$ , and linear with  $A'(u) = 1/400$  elsewhere.

In (a), the constant state on the left  $u_L = 0.6$  is connected to a steady jump at  $x = 0$  by a rarefaction, which is connected to a constant state of approximately  $u = 0.15$ . This constant state is connected to  $u_R = 0.6$  by a shock moving to the right. All of these waves are created by the jump in the coefficient  $\gamma$ . In (b), the shock moving to the right is smaller, due to the diffusion, and the steady jump at  $x = 0$  has been replaced by a shock moving to the left. Both shocks have end states at approximately  $u = 0.45$  and  $u = 0.55$ , providing numerical evidence that discontinuities can only occur in regions of state space where the diffusion  $A(u)$  degenerates to a constant.

**Example 2.** The initial data  $u_0$  and the coefficient  $\gamma$  are the same as in the previous example. The diffusion term is now given by

$$(6.2) \quad A(u) = .0025(u - 0.3) \chi_{[.3, 1]}(u),$$

which is degenerate for  $u < .3$ , and allows for a stationary jump in the solution  $u$  at  $x = 0$ . Figure 2 (a) shows the solution for the purely hyperbolic problem, and (b) shows the effect of adding the diffusion term. The scheme was run for 500 time steps, with  $\Delta x = .02$  and  $\Delta t = .04$ , as in the previous example. The diffusion term has the effect of changing the upper end states for both the steady jump and the shock moving to the right, lowering them to approximately  $u = 0.3$ , providing more numerical evidence that the solution is continuous wherever  $A(u)$  is nondegenerate.

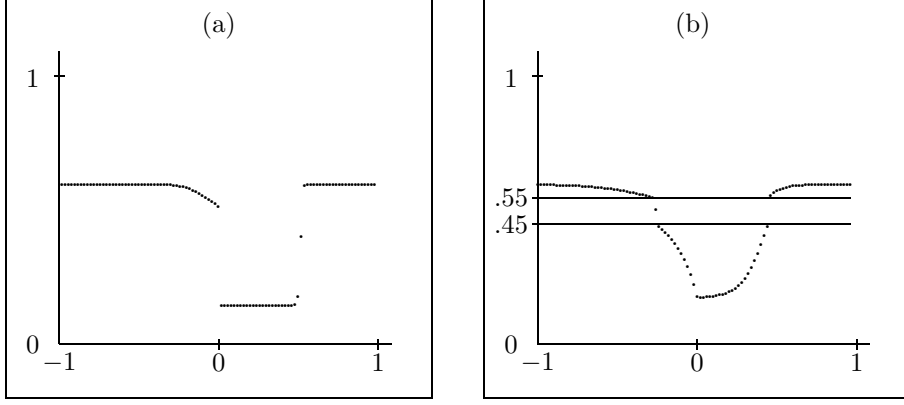


FIGURE 1. Example 1. (a) Purely hyperbolic problem;  $A(u) = 0$ . (b) With degenerate diffusion term (6.1).

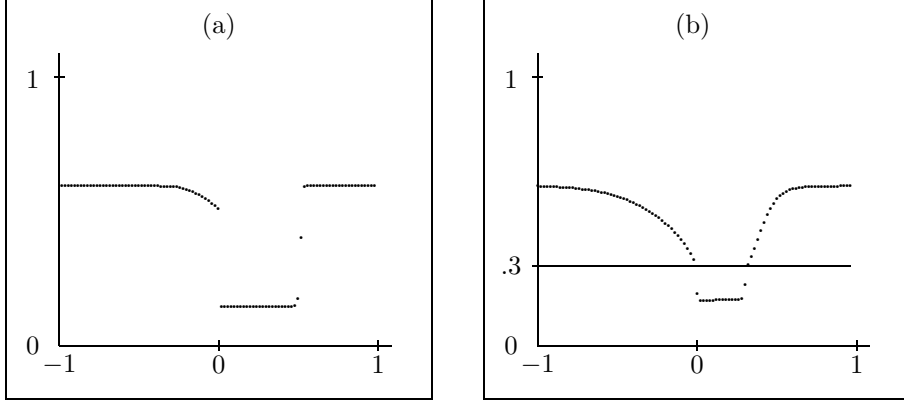


FIGURE 2. Example 2. (a) Purely hyperbolic problem;  $A(u) = 0$ . (b) With the degenerate diffusion term (6.2).

**Example 3.** Figure 3 shows the result of two runs of the scheme (2.4) with the Riemann problem having initial data  $(u_L, u_R) = (0.8, 0.2)$ , and the coefficient given by  $(\gamma_L, \gamma_R) = (0.05, 0.1)$ . Both plots in Figure 3 show the purely hyperbolic case. In (a),  $\Delta x = 0.02$ ,  $\Delta t = 0.04$ , with 100 time steps. In (b),  $\Delta x = 0.01$ ,  $\Delta t = 0.01$ , with 400 time steps. Both plots show a spurious bump that starts out as a kink, and then moves to the right at the edge of the rarefaction. Refining the mesh causes the bump to decrease in amplitude and width, as can be seen by comparing Figure 3 (a) with (b). We have found that these spurious bumps turn up in certain (but not all) Riemann problems. As predicted by our convergence theory, they shrink as the mesh size diminishes. For a Riemann problem the discretizations (2.5) and (2.6) result in an intermediate state  $u_M = (u_L + u_R)/2$  and a sharp jump from  $\gamma_L$  to  $\gamma_R$ . We have found from experience that the bump can be removed by moving the jump in  $\gamma$  one mesh width in the direction of the bump. All of our convergence theory remains valid with a change of this type, since we still have  $\gamma^\Delta \rightarrow \gamma$  in  $L^1_{loc}$  and boundedly a.e. Figure 4 (a) is the same as Figure 3 (a) with the exception that the jump in  $u_0$  has been moved one mesh width to the right, with the result that the spurious bump does not appear. Figure 4 (b) is the same as Figure 4 (a), except that the degenerate diffusion term

$$(6.3) \quad A(u) = .0025 \left( u \chi_{[0,.2]}(u) + .2 \chi_{(.2,1]}(u) \right)$$

is incorporated.

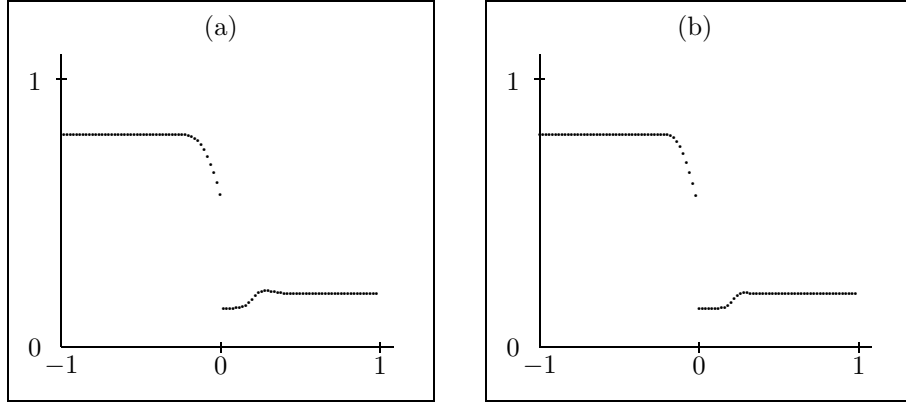


FIGURE 3. Example 3. (a) Purely hyperbolic problem, showing spurious bump;  $A(u) = 0$ . (b) Purely hyperbolic problem. Mesh size reduced to show that the spurious bump reduces in width and amplitude.

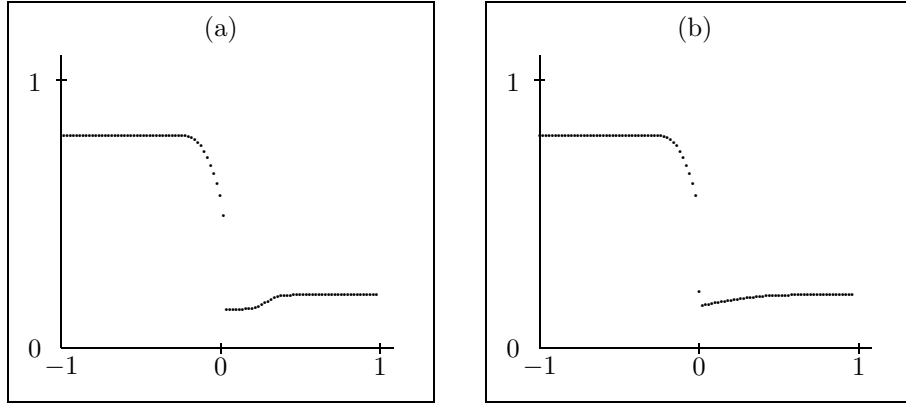


FIGURE 4. Numerical example 3. (a) Purely hyperbolic problem, with jump in  $\gamma$  moved by one mesh width to the right to get rid of spurious bump. (b) With degenerate diffusion term (6.3).

The effect of the diffusion is to smear out the corners where the rarefaction meets the constant states. Although not shown in this plot, when the mesh size is reduced sufficiently, the small jump between the minimum point on the graph and  $u = .2$  fills in, so that the solution is continuous in the region where  $A'(u) > 0$ .

**Example 4.** In this example we study the convergence rate for the problem in the previous example, with the diffusion term included. We used the discretizations (2.5) and (2.6), so that the spurious bump was present, but diminished as the mesh shrank. Table 1. shows the results of the test. The last row in the table was used as the "true" solution;  $L^1$  differences with this solution were computed, and appear in the last column of the table. Although the test is not conclusive, it appears that for this particular example there is linear convergence as  $\Delta x \rightarrow 0$ .

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$\Delta x$	$\Delta t$	Number of steps	$L^1$ error estimate
1/16	1.6	4	0.1973
1/32	0.4	16	0.0137
1/64	0.1	64	0.0105
1/128	0.025	256	0.0042
1/256	0.00625	1024	0.0018
1/512	0.0015625	4096	
Example 4.			

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