Upwind difference schemes for scalar conservation laws with flux interfaces

John Towers

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jtowers@cts.com

Outline

- 0. Introduction
- 1. Concave flux with a discontinuous coefficient
- 2. Nonconvex flux with a discontinuous coefficient and source terms
- 3. Network junctions

0. Introduction

Introduction

- Will discuss simple difference schemes for scalar conservation laws
- One of two types of interface may be present
- 1- Discontinuity in a spatially varying flux parameter
- 2- Network junction
- The design principle: Hide the interfaces at the centers of computational cells
- Allows for standard scalar numerical flux at cell boundaries (will use Godunov, Engquist-Osher)

Introduction

- If no interface present, the theory for both the conservation law and the difference scheme have been well developed for many years

- Difference schemes are monotone (only first order accuracy)

- Much more accurate computational techniques exist for the case with no interface

- If solution to Riemann problem is known at the interface, it can be used to construct a better solution than provided by these schemes

Introduction

- These algorithms are interesting anyway, for several reasons

- 1- Provide a way to proceed if the solution to the Riemann problem is not yet known

- 2- Interesting analytical properties

- 3- Seem to have a built-in selection principle (entropy satisfaction)

- 4- Upwind property gives better resolution than Lax-Friedrichs, which is another approach

1. Concave flux with a discontinuous coefficient

Concave flux - Cauchy problem

- Cauchy problem:

$$u_t + (k(x)f(u))_x = 0, \;\; u(x,0) = u_0(x)$$

- $k \in BV \Rightarrow$ Kruzkov theory does not apply

- $f > 0$, strictly concave, $f(0) = f(1) = 0$. Single maximum $u = u^*$.

- $u_0 \in BV \bigcap L^1$, $u_0 \in [0,1]$

- $0 < \underline{k} \leq k(x) \leq \overline{k}$

Concave flux- as a system

- Possible to view the equation as a non-strictly hyperbolic system

$$u_t + (kf(u))_x = 0, \quad k_t = 0$$

- Simple model of "resonant" system - eigenvalues ($\lambda^k = 0$, $\lambda^u = kf'(u)$) coincide when $f' = 0$

- The equation $u_t + (kf(u))_x$ often used to model more complicated resonant systems

Concave flux-background

- Temple (1982) used Glimm scheme to establish global weak solution for 2×2 resonant system modeling muliphase flow

- First use of the "singular" transformation to establish compactness

- Lin, Temple, Wang (1995): Godunov method for

  $$u_t + f(k(x), u)_x = 0, \quad k_t = 0$$

  modeling resonant system

- Established convergence of Godunov method (singular mapping, again), linear growth rate for $TV(u)$, assuming $TV(u_0)$, $TV(k)$, $TV(k')$ bounded

Concave flux-background

- Klingenberg, Risebro (1995): $u_t + (kf(u))_x = 0$ with conditions on f similar to f concave, $f(0) = f(1) = 0$

- Solved Riemann problem, used solution to construct front tracking algorithm

- Established compactness for approximate solutions using a singular mapping, analysis of wave interactions

- Will use this version of singular mapping for difference scheme

- Limit of front tracking approximations satisfies a ”wave entropy” condition, implying uniqueness

Concave flux - background

- For periodic u_0 and k, the asymptotic solution is a standing N-wave

- Klausen, Risebro (1997): $u_t + f(a(x), u)_x = 0$

- Similar to $u_t + (k(x)f(u))_x = 0$ with f concave and $f(0) = f(1) = 0$

- Solution depends continuously (in $L^1(\mathbf{R})$) on the initial data u_0 and the coefficient k

- First prove for smoothed version of k, then proceed via approximation

Concave flux - background

- Tveito, Winther (1995): Discussed use of 2×2 Lax-Friedrichs scheme for

$$u_t + (k(x)f(u))_x = 0, \quad k_t = 0$$

- No Riemann solvers required (not even scalar) due to central differences

- Motivation was as an alternative approach when either no solution to Riemann problem, or using 2×2 Riemann solvers causes variation blow-up

- Lax-Friedrichs is robust but produces excessive smearing

- Convergence of Lax-Friedrichs (2×2 or scalar version) for $k \in BV$ is an open problem

Scheme for concave flux - the goal

- Devise a ”scalar” difference scheme that does not require knowledge of the solution to the 2×2 Riemann problem

- Preferably an upwind scheme, since shocks are resolved better

- Purpose of the scheme - a tool for understanding solutions of the Cauchy problem, especially the Riemann problem

Scheme for concave flux - the approach

- Place jumps in k at centers of computational cells - away from cell boundaries

- The conserved quantity u and the coefficient are discretized on meshes that are staggered

- Use standard scalar Riemann solvers, eg., scalar Godunov or Engquist-Osher (EO) numerical flux

Scheme for concave flux

- Uniform mesh size Δx on x-axis - $x_j = j\Delta x$, $j \in \{\ldots, -2, -1, 0, +1, +2, \ldots\}$

- $I_j = [x_j - \Delta x/2, x_j + \Delta x/2]$, $\chi_j(x)$ its characteristic function

- $I_{j+\frac{1}{2}} = [x_j, x_{+1}]$, $\chi_{j+\frac{1}{2}}(x)$ its characteristic function

- Uniform spacing of time steps - $t^n = n\Delta t$, $n = 0, 1, \cdots$

- $I^n = [t^n, t^{n+1})$, $\chi^n(t)$ its characteristic function

Scheme for concave flux

- The marching algorithm:

$$u_j^{n+1} = u_j^n - \lambda(k_{j+\frac{1}{2}} h_{j+\frac{1}{2}} - k_{j-\frac{1}{2}} h_{j-\frac{1}{2}})$$

$$h_{j+\frac{1}{2}} = h(u_{j+1}^n, u_j^n)$$

- Discretization of u_0 and k

$$u^{\Delta}(x,0) = \sum_j \chi_j(x) u_j^0, \quad u_j^0 = \frac{1}{\Delta x} \int_{I_j} u_0(x)dx,$$

$$k^{\Delta}(x) = \sum_j \chi_{j+\frac{1}{2}}(x) k_{j+\frac{1}{2}},$$

$$k_{j+\frac{1}{2}} = \frac{1}{\Delta x} \int_{I_{j+\frac{1}{2}}} k(x)dx.$$

Scheme for concave flux - the numerical flux

- Godunov

$$h(v,u) = \begin{cases} \min_{[u,v]} f(w), & \text{if}\, u \leq v \\ \max_{[v,u]} f(w), & \text{if}\, u \geq v \end{cases}$$

- Engquist-Osher (EO)

$$h(v,u) = \frac{1}{2}(f(u) + f(v)) - \frac{1}{2}\int_u^v |f'(w)| dw$$

- Two-point monotone, upwind fluxes

- Observation: Godunov and EO are identical except when u and v are connected by a sonic shock ($u < u^* < v$).

Concave flux - elementary properties of the scheme

- Conservation form

- Scheme is monotone with CFL condition $\lambda \|k\|_\infty \|f'\|_\infty \leq 1$ (check partial derivatives)

- Approximations u_j^n remain in $[0, 1]$ (from monotonicity and $f(0) = f(1) = 0$)

- Time-continuity: $\sum_j |u_j^{n+1} - u_j^n| \leq C$, where C is independent of Δ and n (Crandall-Tartar lemma)

Concave flux - the singular mapping

- The idea - prove compactness for transformed version $z = \Psi(u, k)$ of u

- The approach was first used by Temple (1982)

- The version used by Klingenberg, Risebro (1995) works for the difference schemes:

$$\Psi(u, k) = k\sigma(u - u^*)(1 - f(u)/f^*)$$

Concave flux - properties of the singular function

- Strictly increasing as a function of u, for each fixed k
- $\Psi_u > 0$ everywhere except at $u = u^*$ (''singular'' at u^*)
- For $0 < \underline{k} \leq k \leq \overline{k}$, maps $[0,1] \times [\underline{k}, \overline{k}]$ into $[-\overline{k}, \overline{k}]$
- Lipschitz continuous in both variables

Concave flux - transforming u to z

- Have approximation

$$u^{\Delta}(x,t) = \sum_{n\geq 0} \sum_{j=-\infty}^{+\infty} \chi_j^n(x,t) u_j^n$$

$$\chi_j^n(x,t) = \chi_j(x)\chi^n(t)$$

- $\chi_j(x)$ = characteristic function for $[x_j - \Delta x/2, x_j + \Delta x/2)$, $\chi^n(t)$ = characteristic function for $[t^n, t^n + \Delta t)$

- Transform to $z^{\Delta}(x,t)$ via

$$z^{\Delta}(x,t) = \Psi(u^{\Delta}(x,t), k^{\Delta}(x,t))$$

Concave flux - transforming u to z

- $z^{\Delta}(x,t)$ is piecewise constant. In the x-direction, it has jumps at cell boundaries (due to u^{Δ}) and jumps at cell centers (due to k^{Δ})

- Not hard to check that z^{Δ} inherits L^{∞} bound and time-continuity from u^{Δ}

- For compactness, we need to bound $TV(z^{\Delta})$.

Concave flux - variation bound for z^{Δ}

- Jumps at cell centers (due to k^{Δ}) contribute $constant \times TV(k)$ to $TV(z^{\Delta})$

- We need an estimate of $\sum_j |\Delta_+^u z_j|$, the contribution from cell boundaries (due to jumps in u^{Δ})

- Requires an interesting fact about the Engquist-Osher and Godunov schemes when $k = 1$

Concave flux - Lemma on constant-k EO scheme

- First write the singular function as
$$\Psi(u,k) = \frac{k}{f^*}\int_{u^*}^{u} |f'(w)|dw$$

- Define
$$\phi(u) = \int_{u^*}^{u} |f'(w)|dw$$

- Note that ϕ is strictly increasing

- Notation: $\chi_+(w) = 1$ if $f'(w) > 0$, zero otherwise; χ_- is defined similarly. $f'_+(w) = max(0, f'(w))$; $f'_-(w)$ is defined similarly.

Concave flux - Lemma on constant-k EO scheme

- The lemma:

$$\begin{aligned}(\phi_j - \phi_{j+1})_+ &\le \chi_-(u_j)|h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}| \\ &+ \chi_+(u_{j+1})|h_{j+\frac{3}{2}} - h_{j+\frac{1}{2}}|\end{aligned}$$

- $\chi_-(w) = 1$ if $f'(w) < 0$, and zero otherwise; $\chi_+(w) = 1$ if $f'(w) > 0$, and zero otherwise

- $$\phi_j = \phi(u_j^n) = \int_{u^*}^{u_j^n} |f'(w)|dw$$

Concave flux - Proof of lemma

- Assume $u_j > u_{j+1}$; otherwise proof is immediate

- Then Godunov and EO are identical, so we can use EO flux formulas

- A pair of identities

$$\phi_j - \phi_{j+1} = \int_{u_{j+1}}^{u_j} f'_+(w)dw - \int_{u_{j+1}}^{u_j} f'_-(w)dw$$

$$h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}} = \int_{u_{j-1}}^{u_j} f'_+(w)dw + \int_{u_j}^{u_{j+1}} f'_-(w)dw$$

Concave flux - Proof of lemma

- Suffices to prove the two inequalities

$$\begin{aligned}\chi_-(u_j)\left|\int_{u_{j-1}}^{u_j} f'_+(w)dw\right. &+ \left.\int_{u_j}^{u_{j+1}} f'_-(w)dw\right| \\ &\geq \int_{u_j}^{u_{j+1}} f'_-(w)dw\end{aligned}$$

$$\begin{aligned}\chi_+(u_j)\left|\int_{u_{j-1}}^{u_j} f'_+(w)dw\right. &+ \left.\int_{u_j}^{u_{j+1}} f'_-(w)dw\right| \\ &\geq -\int_{u_{j-1}}^{u_j} f'_+(w)dw\end{aligned}$$

- The rest of the proof consists of checking the various cases, i.e., the location of u_j, u_{j+1}, u_{j-1} with respect to u^*

Concave flux - interpreting the lemma

- Lemma gives information on the evolution of rarefactions

- Substitute $(u_j^n - u_j^{n+1})/\lambda$ for $h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}$

- If $u_j^n > u^*$ and $u_j^n > u_{j+1}^n$, then u_j^n decreases at the next time step, and

$$u_j^{n+1} \leq u_j^n - \lambda \int_{u_j^n}^{u_{j+1}^n} f'_-(w)dw$$

- If $u_j^n < u^*$ and $u_j^n < u_{j-1}^n$, then u_j^n increases at the next time step, and

$$u_j^{n+1} \geq u_j^n - \lambda \int_{u_{j-1}^n}^{u_j^n} f'_+(w)dw$$

Concave flux - the variation bound

- Turns out that the proof is only slightly more difficult with variable k, so set $k = 1$

- With $k = 1$, $\Psi = \phi/f(u^*)$

- Will show that the negative variation of $\phi(u^\Delta(x,t))$ is bounded

- A bound on the total variation then follows since $\phi(u^\Delta(x,t))$ is bounded

Concave flux - the variation bound

- Start from the lemma

$$\begin{aligned}(\phi_j - \phi_{j+1})_+ &\leq \chi_-(u_j)|h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}| \\ &+ \chi_+(u_{j+1})|h_{j+\frac{3}{2}} - h_{j+\frac{1}{2}}|\end{aligned}$$

- Substitute $(u_j^n - u_j^{n+1})/\lambda$ for $h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}$ to get

$$\begin{aligned}(\phi_j - \phi_{j+1})_+ &\leq \chi_-(u_j)|u_j^{n+1} - u_j^n|/\lambda \\ &+ \chi_+(u_{j+1})|u_{j+1}^{n+1} - u_{j+1}^n|/\lambda\end{aligned}$$

Concave flux - the variation bound

- Sum over j, and shift indices to get

$$\sum_j (\phi_j - \phi_{j+1})_+ \leq \sum_j |u_j^{n+1} - u_j^n|/\lambda$$

- The sum on the right is bounded uniformly, basically due to the fact that the scheme is L^1-contractive

- When the coefficient k is properly accounted for, the same bounds result, modulo terms of the form $constant \times TV(k)$

Concave flux - convergence

- Standard compactness arguments (Helly's theorem) are used to show that a subsequence $z^{\Delta} \to z$ in L^1_{loc} and boundedly a.e.

- Define $u = \Psi^{-1}(z, k)$. By continuity of Ψ^{-1}, $u^{\Delta} \to u$ in L^1_{loc} and boundedly a.e.

- The limit u is a weak solution, by the Lax-Wendroff theorem applied to the system

$$u_t + (kf(u))x = 0, \quad k_t = 0$$

Singular funtion, entropy flux, EO flux

- Kruzkov entropy flux associated with entropy $V(u) = |u - u^*|$

  $$F(u) = \sigma(u - u^*)(f(u) - f(u^*)) = -\int_{u^*}^{u} |f'(w)|dw$$

- Singular mapping

  $$\Psi(u, k) = k\sigma(u - u^*)(1 - f(u)/f^*) = -(k/f^*)F(u)$$

- EO flux

  $$h(v, u) = \frac{1}{2}(f(u) + f(v)) - \frac{1}{2}\int_{u}^{v} |f'(w)|dw$$

Concave flux - geometric entropy conditions

- Kruzkov entropy condition for smooth k

$$\int_{\mathbf{R}\times\mathbf{R}^+} (|u-c|\psi_t + k\sigma(u-c)(f(u)-f(c))\psi_x - \sigma(u-c)k'f(c)\psi)dxdt \geq 0$$

- ψ any smooth nonnegative test function with support in $x > 0$, any $c \in \mathbf{R}$

- Not applicable if k is not smooth, but still applicable to this problem away from jumps in k

- Will focus on the Godunov version of the scheme - use the scheme to derive a geometric entropy condition at jump in k

Concave flux - geometric entropy conditions

- Cell entropy for variable-k Godunov scheme derived from constant-k Godunov scheme:

$$\begin{aligned} V_j^{n+1} &\leq V_j^n - \lambda\Delta_+(k_{j-\frac{1}{2}}H_{j-\frac{1}{2}}) \\ &+ (V_j^{n+1} - V(w_j^n)) + \lambda H_{j-\frac{1}{2}}\Delta_+ k_{j-\frac{1}{2}} \end{aligned}$$

- (V, F) any convex entropy pair, $H_{j+\frac{1}{2}} = F(u^G(u_{j+1}^n, u_j^n))$, $V_j^n = V(u_j^n)$

- w_j^n defined by

$$w_j^n = u_j^n - \lambda k_{j+\frac{1}{2}}(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}})$$

Concave flux - geometric entropy conditions

- Using the cell entropy inequality, and a Lax-Wendroff argument, the Kruzkov entropy inequalities are satisfied by limit solution u away from jumps in k

- For piecewise smooth solutions, the usual geometric entropy condition is satisfied for a discontinuity in the smooth-k region

$$kf'(u_L) > s > kf'(u_R)$$

- $s = k(f(u_R) - f(u_L))/(u_R - u_L)$ = shock speed, u_L and u_R are values of u on left and right side of shock

Concave flux - geometric entropy conditions

- Assume finitely many jumps in k, located at ξ_i, $i = 1, \ldots, M$

Now allow ψ to have support including jumps in k, and take $c = u^*$, $f^* = f(u^*)$

$$\int (|u - u^*|\psi_t + k\sigma(u - u^*)(f(u) - f(u^*))\psi_x)dxdt +$$

$$f^* \int (|k'(x)| + \sum_{i=1}^{M} |k(\xi_i^+) - k(\xi_i^-)|\delta(x - \xi_i))\psi dxdt$$

$$\geq 0.$$

- For u smooth on both sides of jump at ξ_i, this entropy inequality yields (with $F(u) = \sigma(u - u^*)(f(u) - f^*)$)

$$k_R F(u_R) - k_L F(u_L) \leq |k_R - k_L| f^*$$

Concave flux - geometric entropy conditions

- The Rankine-Hugoniot condition at a jump in k is $k_L f(u_L) = k_R f(u_R)$

- Lemma: For a pair of states satisfying the Rankine-Hugoniot condition

$$\begin{aligned} k_R F(u_R) - k_L F(u_L) &\leq |k_R - k_L| f(u^*) \\ &\Leftrightarrow \quad f'_-(u_L) f'_+(u_R) = 0. \end{aligned}$$

- $f'_-(u_L) f'_+(u_R) = 0$ means that characteristics on at least one side of the jump in k can be drawn backward to the x-axis

Concave flux - geometric entropy conditions

- Condition E: In smooth k region, characteristics on both sides of the discontinuity extend backward to the x-axis

- Condition E': At jumps in k, characteristics on at least one side of the discontinuity extends backward to the x-axis

- Recall theorem of Quinn (1971) that piecewise smooth solutions of $u_t + f(u)_x = 0$ satisfying condition E form an L^1-contraction semigroup

- Theorem: Assuming k piecewise constant, piecewise smooth solutions of $u_t + (kf(u))_x = 0$ satisfying conditions E and E' form an L^1-contraction semigroup

Concave flux - geometric entropy conditions

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2. Nonconvex flux with a discontinuous coefficient and source terms

Nonconvex flux

- Cauchy problem:

  $$u_t+(k(x)f(u)-a(x))_x = 0, \quad u(x,0) = u_0(x)$$

- f can be nonconvex, but must have $|f'| > 0$ a.e. to allow for inversion of singular mapping

- Allow f to have any finite number of critical points

- Have added a source term $a(x)$, which may also have jumps

- Assume $a, k \in BV$, $\underline{a} \le a(x) \le \overline{a}$, $0 < \underline{k} \le k(x) \le \overline{k}$

Nonconvex - background

- Klingenberg, Risebro (1999): Studied the nonconvex problem

$$\theta_t + (r(x)sin(\theta))_x = 0, \quad r_t = 0$$

- Coefficient satisfies $r > 0$, $r \in BV$

- Assume $\theta_0 \in [-\pi, \pi]$, which is an invariant region

- Problem arose as auxiliary equation to model of 2-phase flow in porous medium

- Constructed solution to the Riemann problem, and used this to devise front tracking scheme

Nonconvex - background

- Compactness via singular mapping and analysis of wave interactions

- Solution operator is L^1-contractive

- First prove this for smoothed coefficient r, then for $r \in BV$ via approximation

Nonconvex - background

- Ostrov (1999): Radar shape from shading (SFS) problem

- Hamilton-Jacobi formulation

$$u_r + g(I(y,r), u_y) = 0$$

- g nonconvex, $I(y,r)$ discontinuous

- Constructed monotone scheme and proved convergence, assuming that certain upper and lower limits are equal

Source terms - background

- Greenberg, Leroux, Baraille, Noussair (1997): Studied $u_t + f(u)_x = a_x$

- f convex, $f(0) = 0$, f even

- Source term a assumed piecewise smooth

- Constructed entropy solution of Riemann problem using minimization of a certain functional as selection principle

- Used this to construct solution to Cauchy problem for a piecewise constant, $u_0 \in L^\infty \bigcap L^1_{loc}$

- Solutions form L^1-contraction semigroup

- Constructed Godunov scheme based on solutions to Riemann problem

Source terms - background

- Established L^∞ stability for Godunov scheme
- Compactness is an open problem

Nonconvex flux - the scheme

- Scheme constructed in the same way, except source terms are present

$$U_j^{n+1} = U_j^n \quad - \quad \lambda((k_{j+\frac{1}{2}} h_{j+\frac{1}{2}} - a_{j+\frac{1}{2}}) \\ \quad - \quad (k_{j-\frac{1}{2}} h_{j-\frac{1}{2}} - a_{j-\frac{1}{2}}))$$

- Discretization of source terms also staggered with respect to discretization of u

$$a_{j+\frac{1}{2}} = \frac{1}{\Delta x} \int_{I_{j+\frac{1}{2}}} a(x)dx$$

- Do not have to know the solution of Riemann problem involving jumps in u_0, k, a - possibly quite complicated

- The flux $h_{j+\frac{1}{2}}$ is EO (Convergence for Godunov version is an open problem)

Nonconvex flux - two things that make this problem harder

- L^∞ bounds require more work for this problem than than previously, where we had $f(0) = f(1) = 0$

- Will use a version of the singular mapping to establish compactness, but the lemma about the EO scheme does not apply here

Nonconvex flux - L^∞ bounds

- Previously assumed $f(0) = f(1) = 0$, which made $[0, 1]$ invariant

- Will allow for other conditions on the flux, basically that the flux is monotonically increasing or decreasing for $|u|$ sufficiently large, and takes on a "sufficiently" large range of values

- "sufficiently" large range of values is determined by $\overline{a} - \underline{a}$ and $\overline{k}/\underline{k}$

Nonconvex flux - L^∞ bounds

- Approach - find fixed points above and below the initial data u_0

- Time advance operator: $\Gamma^\Delta(u^\Delta(\cdot, t^n)) = u^\Delta(\cdot, t^{n+1})$

- $v^\Delta = \sum_{j,n} \chi_j^n(x,t) V_j^n$ is a fixed point if
$$\Gamma^\Delta(v^\Delta) = v^\Delta$$

- Sufficient condition for a fixed point: $\exists$ constant η such that for all j,
$$k_{j+\frac{1}{2}} h(V_{j+1}, V_j) - a_{j+\frac{1}{2}} = \eta$$

Nonconvex flux - L^∞ bounds

- The presence of $h(V_{j+1}, V_j)$ makes this a difficult nonlinear system

- Make the problem easier - require h to be an upwind flux and f be nondecreasing or nonincreasing in the "interval of interest"

- Fixed point condition becomes (if f is non-decreasing): $\exists$ constant η such that for all j,

$$k_{j+\frac{1}{2}} f(V_j) - a_{j+\frac{1}{2}} = \eta$$

- Replace $f(V_j)$ by $f(V_{j+1})$ if f nonincreasing

Nonconvex flux - Lemma on L^∞ bounds

- Assumptions:

- $u_0(x) \in [\tilde{w}, \tilde{v}] \subseteq [\overline{w}, \overline{v}]$, and $\lambda \|k\|_\infty \|f'\|_\infty \leq 1$ for $u \in [\overline{w}, \overline{v}]$

- f is monotone on each of the intervals $[\overline{w}, \tilde{w}]$ and $[\tilde{v}, \overline{v}]$

- There are constants η and ξ such that the following equations have solutions $v \in [\tilde{v}, \overline{v}]$ and $w \in [\overline{w}, \tilde{w}]$ for all $a \in [\underline{a}, \overline{a}]$ and all $k \in [\underline{k}, \overline{k}]$:

$$kf(v) - a = \eta, \quad kf(w) - a = \xi$$

Nonconvex flux - Lemma on L^∞ bounds

- Conclusions:

- The computed approximations $u^\Delta(x,t)$ remain in $[\overline{w}, \overline{v}]$, and the CFL condition continues to be satisfied for $n = 1, 2, \ldots$

- The time advance operator Γ^Δ is monotone on the computed approximations $u^\Delta(x,t)$ (the scheme is monotone)

- The time continuity estimate holds

$$\sum_j |U_j^{n+1} - U_j^n| \leq C$$

Nonconvex flux - L^∞ example 1

- $u_t + f(u)_x = 0$ ($a = 0$, $k = 1$)

$$\widetilde{w} = \overline{w} = \inf_x u_0(x), \quad \widetilde{v} = \overline{v} = \sup_x u_0(x)$$

$$\xi = f(\inf_x u_0(x)), \quad \eta = f(\sup_x u_0(x))$$

$$w(x) = \inf_x u_0(x), \quad v(x) = \sup_x u_0(x)$$

- As expected, lemma yields

$$u^\Delta(x, t^n) \in [\inf_x u_0(x), \sup_x u_0(x)]$$

Nonconvex flux - L^∞ example 2

- $u_t + (kf(u))_x = 0$, $a = 0$, $\underline{k} \leq k(x) \leq \overline{k}$, $f(0) = f(1) = 0$, $u_0(x) \in [0,1]$

$$\widetilde{w} = \overline{w} = 0, \quad \widetilde{v} = \overline{v} = 1$$

$$\xi = 0, \quad \eta = 0$$

$$w(x) = 0, \quad v(x) = 1$$

- As expected, lemma predicts that $[0,1]$ is invariant, another well-known fact

Nonconvex flux - L^∞ example 3

- $u_t + (kf(u))_x = 0$, $a = 0$, $0 < \underline{k} \le k(x) \le \overline{k}$, f strictly convex and even, $f(0) = 0$, eg., $f(u) = u^2/2$

- Let f_l^{-1} and f_r^{-1} denote the negative and positive branches of f^{-1}. The lemma yields:

$$\widetilde{w} = \inf_x u_0(x), \quad \widetilde{v} = \sup_x u_0(x)$$

$$\overline{w} = f_l^{-1}(\frac{\overline{k}}{\underline{k}} f(\widetilde{w})), \quad \overline{v} = f_r^{-1}(\frac{\overline{k}}{\underline{k}} f(\widetilde{v}))$$

$$\xi = \overline{k} f(\inf_x u_0(x)), \quad \eta = \overline{k} f(\sup_x u_0(x))$$

$$w(x) = f_l^{-1}(\frac{\overline{k}}{k(x)} f(\widetilde{w})), \quad v(x) = f_r^{-1}(\frac{\overline{k}}{k(x)} f(\widetilde{v}))$$

Nonconvex flux - L^∞ example 4

- $u_t + (f(u) - a)_x = 0$, $k = 1$, $\underline{a} \le a(x) \le \overline{a}$, f strictly convex and even, $f(0) = 0$, eg., $f(u) = u^2/2$. The lemma produces:

$$\widetilde{w} = \inf_x u_0(x) < 0, \quad \widetilde{v} = \sup_x u_0(x) > 0$$

$$\overline{w} = f_l^{-1}(f(\widetilde{w}) + \overline{a} - \underline{a}), \quad \overline{v} = f_r^{-1}(f(\widetilde{v}) + \overline{a} - \underline{a})$$

$$\xi = f(\widetilde{w}) - \underline{a}, \quad \eta = f(\widetilde{v}) - \underline{a}$$

$$w(x) = f_l^{-1}(f(\widetilde{w}) + a(x) - \underline{a})$$

$$v(x) = f_r^{-1}(f(\widetilde{v}) + a(x) - \underline{a})$$

- Get the following bound, which was already found (in stronger form) by Greenberg, et al (1997) for a Godunov-type scheme for this problem

$$f(u^\Delta(x, t^n)) \le \sup_x f(u_0(x)) + \overline{a} - \underline{a}$$

Nonconvex flux - the singular mapping

- $\Psi(u, k, a) = k \int_0^u |f'(w)| dw - a$

- $\Psi(\cdot, k, a)$ is strictly increasing due to assumption that there are only finitely many critical points

- As in the concave case, map u^Δ to z^Δ via

$$z^\Delta(x,t) = \Psi(u^\Delta(x,t), k^\Delta(x,t), a^\Delta(x,t))$$

- Prove compactness for z^Δ

- L^∞ bound and time-continuity inherited from u^Δ

Nonconvex flux - variation bound for Ψ

- For compactness, require a variation bound on z^{Δ}

- As in the convex case, the parameters k and a only make the proof slightly more involved

- Set $a = 0$, $k = 1$. Will sketch proof of variation bound for $\phi(u) = \int_0^u |f'(w)|dw$, for the case of three critical points

- Preprint 2000-30 on the Conservation Laws Preprint Server (www.math.ntnu.no) contains a proof for general a, k and finitely many critical points

Nonconvex flux - variation bound for ϕ

- Three critical points - assume

$$\overline{w} = u_0^* < u_1^* < u_2^* < u_3^* < u_4^* = \overline{v}$$

- Assume u_1^* is max, u_2^* is min, u_3^* is max

- Decompose ϕ:

$$\phi(u) = \sum_{i=0}^{3} \phi^i(u)$$

- $\phi^i(u) = \phi(u)$ in $[u_i^*, u_{i+1}^*)$, zero elsewhere

- Show that each ϕ_i has bounded variation

Nonconvex flux - variation bound for ϕ

- Kruzkov entropy pairs: $V(u) = |u - c|$, $F(u) = \sigma(u - c)(f(u) - f(c))$

- Cell entropy inequality for EO scheme:

$$V(U_j^{n+1}) \leq V(U_j) - \lambda \Delta_+ H_{j-\frac{1}{2}}$$

- EO numerical entropy flux:

$$\begin{aligned} H_{j+\frac{1}{2}} &= \frac{1}{2}(F(U_j) + F(U_{j+1})) \\ &- \frac{1}{2}\int_{U_j}^{U_{j+1}} \sigma(f'(w))\sigma(w - c) f'(w) dw \end{aligned}$$

- Derived from formula for numerical entropy flux for monotone schemes in Crandall, Majda (1980)

Nonconvex flux - variation bound for ϕ

- A pair of identities for the EO flux

$$\frac{1}{2}(\Delta_+ H_{j-\frac{1}{2}} + \Delta_+ h_{j-\frac{1}{2}}) = \int_{U_j^n}^{U_{j+1}^n} \chi_r(w;c) f'_-(w)dw + \int_{U_{j-1}^n}^{U_j^n} \chi_r(w;c) f'_+(w)dw$$

$$\frac{1}{2}(\Delta_+ H_{j-\frac{1}{2}} - \Delta_+ h_{j-\frac{1}{2}}) = - \int_{U_j^n}^{U_{j+1}^n} \chi_l(w;c) f'_-(w)dw - \int_{U_{j-1}^n}^{U_j^n} \chi_l(w;c) f'_+(w)dw$$

- $\chi_r(w;c) = 1$ if $w \in [c, \infty)$, zero otherwise
  $\chi_l(w;c) = 1$ if $w \in (-\infty, c]$, zero otherwise

Nonconvex flux - variation bound for ϕ

A pair of entropy inequalities:

$$\frac{1}{2}(\Delta_+ H_{j+\frac{1}{2}} + \Delta_+ h_{j+\frac{1}{2}}) \leq \frac{1}{\lambda}(U_j^n - U_j^{n+1})_+$$

$$\frac{1}{2}(\Delta_+ H_{j+\frac{1}{2}} - \Delta_+ h_{j+\frac{1}{2}}) \leq \frac{-1}{\lambda}(U_j^n - U_j^{n+1})_-$$

Proof:

$$\Delta_+ H_{j+\frac{1}{2}} \leq \frac{1}{\lambda}|U_j^n - U_j^{n+1}|$$

$$\Delta_+ h_{j+\frac{1}{2}} = \frac{1}{\lambda}(U_j^n - U_j^{n+1})$$

Add and divide by 2, subtract and divide by 2.

Nonconvex flux - variation bound for ϕ

- Another pair of entropy inequalities:

$$\int_{U_j^n}^{U_{j+1}^n} \chi_r(w;c) f_-'(w)dw +$$
$$\int_{U_{j-1}^n}^{U_j^n} \chi_r(w;c) f_+'(w)dw \leq \frac{1}{\lambda}(U_j^n - U_j^{n+1})_+$$

$$-\int_{U_j^n}^{U_{j+1}^n} \chi_l(w;c) f_-'(w)dw$$
$$-\int_{U_{j-1}^n}^{U_j^n} \chi_l(w;c) f_+'(w)dw \leq \frac{-1}{\lambda}(U_j^n - U_j^{n+1})_-$$

- Proof: Compare the pair of identities with the previous pair of entropy inequalities

Nonconvex flux - variation bound for ϕ

- Take the ψ_l entropy inequality with $c = u_1^*$ - the only contribution is from $[u_0^*, u_1^*]$

$$-\int_{U_{j-1}}^{U_j} \chi_l(w, u_1^*) f'_+(w) \leq \frac{-1}{\lambda}(U_j^n - U_j^{n+1})_-$$

- This yields

$$(\phi^0(U_{j-1}) - \phi^0(U_j))_+ \leq \frac{1}{\lambda}|U_j^n - U_j^{n+1}|$$

- Sum over j to get a bound on the negative variation of ϕ^0

- $TV(\phi^0) \leq C$, since ϕ^0 is bounded

Nonconvex flux - variation bound for ϕ

- Proceed this way with the χ_l inequalities using $c = u_2^*$, $c = u_3^*$ - get 2 more inequalities,

$$
\begin{aligned}
(\phi^1(U_{j+1}) - \phi^1(U_j)) &- (\phi^0(U_j) - \phi^0(U_{j-1})) \\
&\leq \frac{1}{\lambda}|U_j^n - U_j^{n+1}|
\end{aligned}
$$

$$
\begin{aligned}
(\phi^1(U_{j+1}) - \phi^1(U_j)) &- (\phi^0(U_j) - \phi^0(U_{j-1})) \\
&- (\phi^2(U_j) - \phi^2(U_{j-1})) \\
&\leq \frac{1}{\lambda}|U_j^n - U_j^{n+1}|
\end{aligned}
$$

- The first inequality, along with bound on $TV(\phi^0)$ gives bound on $TV(\phi^1)$, similarly for $TV(\phi^2)$ using the second inequality

Nonconvex flux - variation bound for ϕ

- For ϕ^3, use the χ_r inequality with $c = u_3^*$

$$\phi^3(U_j) - \phi^3(U_{j-1}) \leq \frac{1}{\lambda}|U_j^n - U_j^{n+1}|$$

- This gives a bound on $TV(\phi^3)$

Nonconvex flux - variation bound for Ψ

- When k and a are accounted for, the same bounds result modulo terms of the form

$$C_1 TV(k) + C_2 TV(a)$$

Maximum and minimum principles for the flux

- Define $F^n_{j+\frac{1}{2}} = k_{j+\frac{1}{2}} h^n_{j+\frac{1}{2}} - a_{j+\frac{1}{2}}$, and impose the more restrictive CFL condition $\lambda\|k\|\|f'\| \leq 1/2$

- Max and min principles for the numerical flux:

$$\min_j F^0_{j+\frac{1}{2}} \leq F^n_{j+\frac{1}{2}} \leq \max_j F^0_{j+\frac{1}{2}}$$

- Let $\Delta \to 0$. Max and min principles for the limit of the difference scheme

$$\begin{aligned}&\inf_x (k(x) f(u_0(x)) - a(x)) \\ &\leq k(x) f(u(x,t)) - a(x) \leq \\ &\sup_x (k(x) f(u_0(x)) - a(x))\end{aligned}$$

Maximum and minimum principles for the flux

- Greenberg, et. al. (1997) established these principles for $u_t + f(u)_x = a_x$ and their Godunov scheme with f convex

- Resulted from their construction of the Riemann problem

- CFL: $\lambda \|f'\| \leq 1$

Max and min principles for the flux

- Write the numerical flux in incremental form:

$$F^{n+1}_{j+\frac{1}{2}} = F^n_{j+\frac{1}{2}} + C^+_{j+\frac{1}{2}} \Delta_+ F^n_{j+\frac{1}{2}} - C^-_{j-\frac{1}{2}} \Delta_- F^n_{j+\frac{1}{2}}$$

- Incremental coefficients ($D_{j+\frac{1}{2}} = \Delta_+ F^n_{j+\frac{1}{2}}$)

$$C^+_{j+\frac{1}{2}} = -\lambda k_{j+\frac{1}{2}} \int_0^1 f'_-(U^n_{j+1} - \theta D_{j+\frac{1}{2}}) d\theta$$

$$C^-_{j-\frac{1}{2}} = +\lambda k_{j+\frac{1}{2}} \int_0^1 f'_+(U^n_j - \theta D_{j-\frac{1}{2}}) d\theta$$

- $\Rightarrow F^{n+1}_{j+\frac{1}{2}} \in co\{F^n_{j-\frac{1}{2}}, F^n_{j+\frac{1}{2}}, F^n_{j+\frac{3}{2}}\}$

Nonconvex - entropy conditions

- Entropy inequality at a jump in k and a: For all $c \in \mathbf{R}$

$$k_R F_R - k_L F_L \leq |f(c)(k_R - k_L) - (a_R - a_L)|$$

- $F(u) = \sigma(u-c)(f(u) - f(c))$, $V(u) = |u-c|$

- Derived from the cell entropy for the scheme

$$\begin{aligned} V_j^{n+1} &\leq V_j^n - \lambda\Delta_+(k_{j-\frac{1}{2}}H_{j-\frac{1}{2}}) \\ &+ \lambda|f(c)\Delta_+ k_{j-\frac{1}{2}} - \Delta_+ a_{j-\frac{1}{2}}| \end{aligned}$$

- $H_{j-\frac{1}{2}} =$ Crandall-Majda numerical entropy flux

Nonconvex - entropy conditions

- Take $f \geq 0$, $a = 0$, $k \geq \underline{k} > 0$. Let $c \in co\{u_L, u_R\}$

- $k_L \leq k_R, u_L \leq u_R \Rightarrow f_R \leq f(c)$

- $k_L \leq k_R, u_L \geq u_R \Rightarrow f_L \geq f(c)$

- $k_L \geq k_R, u_L \leq u_R \Rightarrow f_L \leq f(c)$

- $k_L \geq k_R, u_L \geq u_R \Rightarrow f_R \geq f(c)$

- When $k_L = k_R$, recover the usual geometric entropy conditions

3. Network junctions

Network junctions - background

- Problem comes from traffic flow on a network of one-directional roads

- Subject of conservation laws on a network was initiated by Holden, Risebro (1995) [HR95]

- The scalar conservation law $\rho_t + f(\rho)_x = 0$ holds on each "edge" of a finite directed graph

- f concave, $f(0) = f(1) = 0$, with a unique single max at $\sigma \in [0, 1]$ (standard traffic flow modeling)

- Vertices of the directed graph correspond to junctions

Network junctions - background

- Requires extension of notion of weak solution

- Weak solution [HR95]

$$\sum_{i=1}^{N}\left(\int_0^\infty \int_{a_i}^{b_i}(\rho_i \frac{\partial \phi_i}{\partial t} + f(\rho_i)\frac{\partial \phi_i}{\partial x})dxdt\right.$$
$$+ \left.\int_{a_i}^{b_i} \rho_{i,0}(x)\phi_i(x,0)dx\right) = 0$$

  for $\{\phi_i\}_1^N$ any collection of smooth test functions with ϕ_i defined on $[a_i, b_i]$, compactly supported

- Require that the ϕ_i meet C^1 smoothly at junctions

Network junctions - background

- Rankine-Hugoniot condition at a junction J [HR95]

$$\sum_{i=1}^{n} f(\rho_i(b_i,\cdot)) = \sum_{i=n+1}^{n+m} f(\rho_i(a_i,\cdot))$$

- At this junction J there are $N = n + m$ edges, the first n incoming, the last m outgoing

- The edges joining at J have endpoints $[a_i, b_i]$ with $b_i = c_i$ for an imcoming edge, $a_i = c_i$ for an outgoing edge, c_i being the junction point

Network junctions - background

- Rankine-Hugoniot condition not enough to single out unique solution at a junction

- Approach of Holden, Risebro: Maximize the flow at J, measured by
$$\sum_{i=1}^{N} g(f(\rho_i)/f_{max})$$
subject to the Rankine-Hugoniot condition

- g is strictly concave

- Theorem [RH95]: The Riemann problem for a junction has a unique solution under these conditions

Network junctions - background

- The solution of the Riemann problem was used to construct a front tracking algorithm

- Theorem [HR95]: The front tracking algorithm converges to a weak solution of the Cauchy problem

- Compactness established using a version of the singular mapping, and analysis of wave interactions

Network junctions - difference scheme approach

- Plan is to study this problem from the point of view of upwind difference schemes such as Godunov and EO

- Will hide the junctions at the centers of computational cells - finite volume approach

- What does the scheme produce as the solution to the Riemann problem?

- Can we characterize this solution? If so, how does it relate to the maximal flow solution of [HR95]?

- Does the scheme converge to a weak solution?

Network junctions - difference scheme approach

- Will describe approach with a single-junction network, 2 incoming edges, 2 outgoing

- Incoming variables $u(x,t)$, $v(x,t)$, each defined on $(-\infty, 0)$.

- Outgoing variables $w(x,t)$, $z(x,t)$, each defined on $(0, +\infty)$.

- Junction variable $c(t)$, associated with $x = 0$.

Network junctions - Discretize the network

- Spatial mesh width $= \Delta x$

- Outgoing edges - cell centers at $x_j = j\Delta x$, $j = 1, 2, \ldots$

- Incoming edges - cell centers at $x_j = j\Delta x$, $j = -1, -2, \ldots$

- That leaves an interval of length $\Delta x/2$ at the beginning (end) of each outgoing (incoming) edge, which will belong to the junction cell

Network junctions - Discretize the network

- The junction cell now has total measure $4 \times \Delta x/2 = 2\Delta x$

- For a more general junction, the total measure of the junction cell will be $N\Delta x/2$, where N is the total number of edges meeting at the junction J

- Let $\lambda = \Delta x/\Delta t$

- Let U_j^n, V_j^n $(j \leq -1)$ be the incoming approximations, W_j^n, Z_j^n $(j \geq 1)$ the outgoing, and C^n the approximation for the junction cell

Network junctions - the difference equations

- Outgoing edge (eg., the W edge):

  $$W_j^{n+1} = W_j^n - \lambda\Delta_+ h(W_j^n, W_{j-1}^n), \quad j \geq 1$$

- Use $W_0^n = C^n$ in this formula

- Incoming edge (eg., the U edge):

  $$U_j^{n+1} = U_j^n - \lambda\Delta_- h(U_{j+1}^n, U_j^n), \quad j \leq -1$$

- Use $U_0^n = C^n$

- Take $h =$ Godunov or EO flux consistent with f, but other choices of 2-point numerical fluxes make sense

Network junctions - the difference equations

- Difference equation for the junction cell

$$\begin{aligned} C^{n+1} = C^n \;&-\; \frac{\Delta t}{2\Delta x}(h(W_1^n, C^n) + h(Z_1^n, C^n) \\ &-\; h(C^n, U_{-1}^n) - h(C^n, V_{-1}^n)) \end{aligned}$$

- This approach works for more general junctions. Let $\mu^{\Delta}(J)$ = measure of junction cell $= N\Delta x/2$

$$C^{n+1} = C^n - \frac{\Delta t}{\mu^{\Delta}(J)}\left(\sum_{i=n+1}^{n+m} h^i_{\frac{1}{2}} - \sum_{i=1}^{n} h^i_{-\frac{1}{2}}\right)$$

- Similar to finite volume algorithm

Network junctions - difference scheme properties

- Scheme designed to be conservative - preserves the integral over the network=Ω with the natural measure on the network

$$\int_\Omega \rho^\Delta(x, t^{n+1})d\mu = \int_\Omega \rho^\Delta(x, t^n)d\mu$$

- Scheme is monotone with the CFL condition $\lambda\|f'\| \leq 1$. Eg., check (with EO flux)

$$\frac{\partial C^{n+1}}{\partial C^n} = 1 - \frac{\Delta t}{2\Delta x}(2f'_+(C^n) - 2f'_-(C^n))$$

- If number of incoming edges $\neq$ number of outgoing, a similar calculation shows that we get a more restrictive CFL condition

Network junctions - difference scheme properties

- Monotonicity and $f(0) = f(1) = 0 \Rightarrow$ solutions remain in $[0, 1]$

- Monotonicity + preserves integral $\Rightarrow$ time continuity (Crandall-Tartar lemma)
$$\int_{\Omega} |\rho^{\Delta}(x, t^{n+1}) - \rho^{\Delta}(x, t^{n})| d\mu \leq C$$
where C depends on the the total variation of the initial data

- Using this bound, we can get a bound on the total variation of $\Psi(\rho^{\Delta})$, yielding compactness for the computed approximations

Network junctions - Numerical experiments

- Currently trying to derive entropy condition based on output of this scheme (Godunov version) for Riemann problems

- Shocks smeared more than would occur in the usual (no junction) situation

- Have some numerical evidence that the scheme has a flow maximization mechanism built in